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Analytic and numeric computation of two dimensional unsteady nonlinear coupled viscous generalized Burgers' equation

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ABSTRACT

In this paper, we derive a general analytical solution of the two dimensional coupled unsteady nonlinear generalized burgers' equations via Hopf-Cole transformation and separation of variable method, which is used to generate three different sets of initial and boundary conditions. These sets of conditions are used for numerical approximations of the Burgers' equations using the described implicit Logarithmic finite-difference method on the uniform mesh point. The effect of the variation in the Reynolds number is investigated and the accuracy of the scheme is determined through error norms. The described scheme is stable for the small time step and high Reynolds number.

Keywords: Burgers' equations; Hopf-Cole transformation; Separation of variable; Logarithmic Finite-difference method; Reynolds number

1. Introduction

Navier-Stokes equation is the fundamental equation for the description of complex fluid flow, for which the full solution is still extremely difficult in the full domain of physical interest. The coupled Burgers' equations are an appropriate form of the Navier-Stokes equations. They have the same convective and diffusion form as the incompressible Navier-Stokes equations. It is an important simple model for the understanding of physical flows and problems, such as

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hydrodynamic turbulence, shock wave theory, wave processes in thermo-elastic medium, vorticity transport, transport and dispersion of pollutants in rivers, and sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity [1-3]. Simulation of Burgers' equation is a natural and first step towards developing methods for the computation of complex flows. In the past a few decades, it becomes customary to test new approaches in computational fluid dynamics by applying them to Burgers' equation.

Analytic solution of coupled Burgers' equations was first proposed by Fletcher [4] using the Hopf-Cole transformation which he used to generate initial and boundary conditions. Many other researchers also obtained exact solution for two and (2+1) dimensional Burgers' equations [5-14]. The numerical solution of coupled Burgers' equations has been solved by many researchers. Jain and Holla [15] developed two algorithms based on cubic spline method. Fletcher [16] discussed the comparison of a number of different numerical approaches. Wubs and Goede [17] applied an explicit implicit method. Goyon [18] used several multilevel schemes with ADI. Bahadir [19] has applied a fully implicit method used two different sets of initial and boundary conditions. Srivastava *et al.* [20] used the same sets of conditions to examine the accuracy of Crank-Nicolson method. Tamsir and Srivastava [21] developed a semi-implicit finite-difference approach for Burgers' equation, while Srivastava and Tamsir [22] developed Crank-Nicolson semi implicit scheme for 2D Burger's equations. Some more recent works related with Burgers' equation can be referred in [23-28].

The purpose of this paper to derive a general analytical solution of two dimensional coupled Burgers' equations, which is used to generate three different sets of initial and boundary conditions, which are then implemented for the numerical approximations of the Burgers' equations through the described implicit Logarithmic finite-difference method [26].

2. Governing equations and generating exact solution

Consider the two dimensional unsteady nonlinear generalized coupled Burgers' equations of the form:

$$u_t + \mu\left(uu_x + vu_y\right) = \kappa\left(u_{xx} + u_{yy}\right), \tag{1}$$

$$v_t + \mu\left(uv_x + vv_y\right) = \kappa\left(v_{xx} + v_{yy}\right), \tag{2}$$

subject to the initial conditions

$$\left.\begin{aligned} u(x,y,0) &= a_1(x,y), \\ v(x,y,0) &= a_2(x,y), \end{aligned}\right\} \quad (x,y)\in\Omega, \tag{3}$$

and the boundary conditions

$$\left.\begin{aligned} u(x,y,t) &= b_1(x,y,t), \\ v(x,y,t) &= b_2(x,y,t), \end{aligned}\right\} \quad (x,y)\in\Omega, \quad t>0, \tag{4}$$

where $\Omega = \{(x, y) : a \le x \le b,\ c \le y \le d\}$ is the computational domain and $\partial\Omega$ is its boundary; μ and κ are arbitrary positive constants; $u(x, y, t)$ and $v(x, y, t)$ are the velocity components to be determined; a_1, a_2, b_1 and b_2 are the known functions; u_t is unsteady term; uu_x is the nonlinear convection term; $\kappa(u_{xx} + u_{yy})$ is the diffusion term.

An analytical solution of the two dimensional generalized coupled Burgers' equation can be generated via Hopf-Cole transformation [4]

$$u = \left(\frac{-2\kappa}{\mu}\right)\frac{\phi_x}{\phi}, \tag{5}$$

$$v = \left(\frac{-2\kappa}{\mu}\right)\frac{\phi_y}{\phi}. \tag{6}$$

Also let

$$u = f_1(\phi),\ v = f_2(\phi), \tag{7}$$

then from Eqs. (1) and (7), we get

$$f_1'(\phi)\phi_t + \mu\Big(f_1(\phi)\,f_1'(\phi)\phi_x + f_2(\phi)\,f_1'(\phi)\phi_y\Big) = \kappa\Big[f_1''(\phi)\phi_x^{\,2} + f_1'(\phi)\phi_{xx} + f_1''(\phi)\phi_y^{\,2} + f_1'(\phi)\phi_{yy}\Big], \tag{8}$$

and from Eqs. (2) and (7), we get

$$f_2'(\phi)\phi_t + \mu\Big(f_1(\phi)\,f_2'(\phi)\phi_x + f_2(\phi)\,f_2'(\phi)\phi_y\Big) = \kappa\Big[f_2''(\phi)\phi_x^{\,2} + f_2'(\phi)\phi_{xx} + f_2''(\phi)\phi_y^{\,2} + f_2'(\phi)\phi_{yy}\Big]. \tag{9}$$

We suppose that ϕ is bounded so that $f_1'(\phi)$ and $f_2'(\phi)$ are all nonzero functions. Taking any of above Eqs. (8) and (9), the same solution is obtained, so we take any one equation, say Eq. (8), divide it by $f_1'(\phi)$ both sides, we get

$$\phi_t + \mu\big(f_1(\phi)\phi_x + f_2(\phi)\phi_y\big) = \kappa\left[\left(\frac{f_1''(\phi)}{f_1'(\phi)}\right)\left(\phi_x^{\,2} + \phi_y^{\,2}\right) + \left(\phi_{xx} + \phi_{yy}\right)\right]. \tag{10}$$

Now since,

$$u = f_1(\phi) = \left(\frac{-2\kappa}{\mu}\right)\frac{\phi_x}{\phi}$$

$$\Rightarrow f_1'(\phi) = \left(\frac{2\kappa}{\mu}\right)\frac{\phi_x}{\phi^2}$$

$$\Rightarrow f_1''(\phi) = \left(\frac{-4\kappa}{\mu}\right)\frac{\phi_x}{\phi^3}$$

$$\Rightarrow \frac{f_1''(\phi)}{f_1'(\phi)} = \frac{-2}{\phi}, \tag{11}$$

plugging Eqs. (10) and (11), we see that ϕ is the solution of the equation:

$$\phi_t = \kappa\left(\phi_{xx} + \phi_{yy}\right). \tag{12}$$

Eq. (12) is linear equation (two dimensional heat equations) which we solve by separation of variable method which is when transformed back into Eqs. (5) and (6) gives the desired analytical solutions u and v of the coupled Burgers' equations (1) and (2).

Consider a general solution of Eq. (12), of the form

$$\phi(x,y,t) = a_1 + a_2 x + a_3 y + a_4 xy + X(x)Y(y)T(t), \tag{13}$$

which is the sum of the bilinear solution $\phi_1(x,y,t) = a_1 + a_2 x + a_3 y + a_4 xy$ and the separable solution $\phi_2(x,y,t) = X(x)Y(y)T(t)$, where a_1, a_2, a_3 and a_4 are arbitrary constants.

The separable solution $\phi_2(x,y,t)$ can be written as

$$\phi_2(x,y,t) = X(x)Y(y)T(t) = Z(x,y)T(t), \tag{14}$$

since ϕ_2 is a solution of Eq. (12), replacing ϕ_2 value from Eq. (14) into Eq. (12), we get

$$\mathrm{ZT}' = \kappa\left(Z_{xx}T + Z_{yy}T\right), \tag{15}$$

or

$$\frac{ZT'}{\kappa} = (\Delta Z)T, \tag{16}$$

where $\mathrm{T}' = \frac{\partial \mathrm{T}}{\partial \mathrm{t}}$ and $\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the two dimensional Laplace operator.

Eq. (16) can be rearranged as

$$\frac{1}{\kappa}\left(\frac{\mathrm{T}'}{\mathrm{T}}\right) = \frac{\Delta Z}{Z} = -\alpha^2, \tag{17}$$

where α^2 is a separation constant and negative sign is used because a decaying function of time is anticipated. Eq. (17) gives two separated equations

$$T' + \alpha^2 \kappa T = 0, \tag{18}$$

$$\Delta Z + \alpha^2 Z = 0. \tag{19}$$

Eq. (18) yields

$$T(t) = Ae^{-\alpha^2 \kappa t}. \tag{20}$$

Eq. (19) can be expressed as

$$X''Y + XY'' + \alpha^2 XY = 0, \tag{21}$$

or

$$\frac{X''}{X} = -\frac{Y''}{Y} - \alpha^2 = -\beta^2, \tag{22}$$

where β^2 is another separation constant. Eq. (22) gives two equations

$$X'' + \beta^2 X = 0, \tag{23}$$

$$Y'' + \gamma^2 Y = 0, \tag{24}$$

where $\gamma^2 = \alpha^2 - \beta^2$. The solutions of Eqs. (23) and (24) are given by

$$X(x) = B\sin\beta x + C\cos\beta x, \tag{25}$$

$$Y(y) = D\sin\gamma y + E\cos\gamma y, \tag{26}$$

where B, C, D, E are arbitrary constants.

Hence the general solution $\phi(x, y, t)$ becomes

$$\phi(x, y, t) = a_1 + a_2 x + a_3 y + a_4 xy + (B\sin\beta x + C\cos\beta x)(D\sin\gamma y + E\cos\gamma y)Ae^{(-\alpha^2\kappa)t}. \tag{27}$$

From Eqs. (5) and (6) we obtain the desired analytical solutions u and v of the generalized two dimensional coupled Burgers' equations given by:

$$u = \left(\frac{-2\kappa}{\mu}\right)\left(\frac{a_2 + a_4 y + A\beta(B\cos\beta x - C\sin\beta x)(D\sin\gamma y + E\cos\gamma y)e^{-\alpha^2\kappa t}}{a_1 + a_2 x + a_3 y + a_4 xy + A(B\sin\beta x + C\cos\beta x)(D\sin\gamma y + E\cos\gamma y)e^{-\alpha^2\kappa t}}\right), \tag{28}$$

$$v = \left(\frac{-2\kappa}{\mu}\right)\left(\frac{a_3 + a_4 x + A\gamma(B\sin\beta x + C\cos\beta x)(D\cos\gamma y - E\sin\gamma y)e^{-\alpha^2\kappa t}}{a_1 + a_2 x + a_3 y + a_4 xy + A(B\sin\beta x + C\cos\beta x)(D\sin\gamma y + E\cos\gamma y)e^{-\alpha^2\kappa t}}\right). \tag{29}$$

We notice that if $\mu=1$ and $\kappa = \upsilon = \frac{1}{\text{Re}}$; υ is the viscosity and Re is the Reynolds number, then Eqs. (1) and (2) take the form:

$$u_t + uu_x + vu_y = \frac{1}{\text{Re}}(u_{xx} + u_{yy}), \tag{30}$$

$$v_t + uv_x + vv_y = \frac{1}{\text{Re}}(v_{xx} + v_{yy}), \tag{31}$$

which is the most popular two dimensional nonlinear coupled Burgers' equation, whose solutions are given by:

$$u = \left(\frac{-2}{\text{Re}}\right)\left(\frac{a_2 + a_4 y + A\beta(B\cos\beta x - C\sin\beta x)(D\sin\gamma y + E\cos\gamma y)e^{\left(\frac{-\alpha^2}{\text{Re}}\right)t}}{a_1 + a_2 x + a_3 y + a_4 xy + A(B\sin\beta x + C\cos\beta x)(D\sin\gamma y + E\cos\gamma y)e^{\left(\frac{-\alpha^2}{\text{Re}}\right)t}}\right), \tag{32}$$

$$v = \left(\frac{-2}{\text{Re}}\right)\left(\frac{a_3 + a_4 x + A\gamma(B\sin\beta x + C\cos\beta x)(D\cos\gamma y - E\sin\gamma y)e^{\left(\frac{-\alpha^2}{\text{Re}}\right)t}}{a_1 + a_2 x + a_3 y + a_4 xy + A(B\sin\beta x + C\cos\beta x)(D\sin\gamma y + E\cos\gamma y)e^{\left(\frac{-\alpha^2}{\text{Re}}\right)t}}\right). \tag{33}$$

3. Implicit Logarithmic Finite-difference scheme for Burgers' Equations

Let $A(u)$ and $B(u)$ be any two continuous differential functions. Multiplying Eqs. (1) and (2) by the derivatives of A and B, respectively yield to the following equations

$$\frac{\partial A}{\partial u}\frac{\partial u}{\partial t} = -A'(u)\left[\mu\left(uu_x + vu_y\right) - \kappa\left(u_{xx} + u_{yy}\right)\right], \tag{34}$$

$$\frac{\partial B}{\partial v}\frac{\partial v}{\partial t} = -B'(v)\left[\mu\left(uv_x + vv_y\right) - \kappa\left(v_{xx} + v_{yy}\right)\right]. \tag{35}$$

Eqs. (34) and (35) can be written as

$$\frac{\partial A}{\partial t} = -A'(u)\left[\mu\left(uu_x + vu_y\right) - \kappa\left(u_{xx} + u_{yy}\right)\right], \tag{36}$$

$$\frac{\partial B}{\partial t} = -B'(v)\left[\mu\left(uv_x + vv_y\right) - \kappa\left(v_{xx} + v_{yy}\right)\right]. \tag{37}$$

Using the usual forward difference method for $\left(\frac{\partial A}{\partial t}, \frac{\partial B}{\partial t}\right)$ and implicit central finite-differences to the convection and diffusion terms of Eqs. (36) and (37), we obtain the finite-difference representations as

$$A\left(u_{i,j}^{n+1}\right) = A\left(u_{i,j}^{n}\right) - \Delta t A'\left(u_{i,j}^{n}\right)\left[\mu\left(u_{i,j}^{n+1}\left(u_x\right)_{i,j}^{n+1} + v_{i,j}^{n+1}\left(u_y\right)_{i,j}^{n+1}\right) - \kappa\left\{\left(u_{xx}\right)_{i,j}^{n+1} + \left(u_{yy}\right)_{i,j}^{n+1}\right\}\right], \tag{38}$$

$$B\left(v_{i,j}^{n+1}\right) = B\left(v_{i,j}^{n}\right) - \Delta t B'\left(v_{i,j}^{n}\right)\left[\mu\left(u_{i,j}^{n+1}\left(v_x\right)_{i,j}^{n+1} + v_{i,j}^{n+1}\left(v_y\right)_{i,j}^{n+1}\right) - \kappa\left\{\left(v_{xx}\right)_{i,j}^{n+1} + \left(v_{yy}\right)_{i,j}^{n+1}\right\}\right]. \tag{39}$$

Now if we let $A(u) = e^u$ and $B(v) = e^v$, then we obtain the Logarithmic finite-difference scheme [26]

$$u_{i,j}^{n+1} = u_{i,j}^{n} - \log_e\left[1 - \Delta t\left\{\mu\left(u_{i,j}^{n+1}\left(u_x\right)_{i,j}^{n+1} + v_{i,j}^{n+1}\left(u_y\right)_{i,j}^{n+1}\right) - \kappa\left(\left(u_{xx}\right)_{i,j}^{n+1} + \left(u_{yy}\right)_{i,j}^{n+1}\right)\right\}\right], \tag{40}$$

$$v_{i,j}^{n+1} = v_{i,j}^{n} - \log_e\left[1 - \Delta t\left\{\mu\left(u_{i,j}^{n+1}\left(v_x\right)_{i,j}^{n+1} + v_{i,j}^{n+1}\left(v_y\right)_{i,j}^{n+1}\right) - \kappa\left(\left(v_{xx}\right)_{i,j}^{n+1} + \left(v_{yy}\right)_{i,j}^{n+1}\right)\right\}\right]. \tag{41}$$

The finite-differences for the derivatives are given by

$$\left.\begin{aligned}
&\left(u_x\right)_{i,j}^{n+1} = \frac{u_{i+1,j}^{n+1} - u_{i-1,j}^{n+1}}{2\Delta x};\ \left(u_y\right)_{i,j}^{n+1} = \frac{u_{i,j+1}^{n+1} - u_{i,j-1}^{n+1}}{2\Delta y};\\
&\left(v_x\right)_{i,j}^{n+1} = \frac{v_{i+1,j}^{n+1} - v_{i-1,j}^{n+1}}{2\Delta x};\ \left(v_y\right)_{i,j}^{n+1} = \frac{v_{i,j+1}^{n+1} - v_{i,j-1}^{n+1}}{2\Delta y};\\
&\left(u_{xx}\right)_{i,j}^{n+1} = \frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{\left(\Delta x\right)^2};\ \left(u_{yy}\right)_{i,j}^{n+1} = \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{\left(\Delta y\right)^2};\\
&\left(v_{xx}\right)_{i,j}^{n+1} = \frac{v_{i+1,j}^{n+1} - 2v_{i,j}^{n+1} + v_{i-1,j}^{n+1}}{\left(\Delta x\right)^2};\ \left(v_{yy}\right)_{i,j}^{n+1} = \frac{v_{i,j+1}^{n+1} - 2v_{i,j}^{n+1} + v_{i,j-1}^{n+1}}{\left(\Delta y\right)^2};
\end{aligned}\right\}, \tag{42}$$

where $u_{i,j}^{n}$ and $v_{i,j}^{n}$ denote the discrete approximations of $u(x,y,t)$ and $v(x,y,t)$ respectively at the mesh point $(i\Delta x, j\Delta y, n\Delta t)$, $\left(i = 0,1,2\ldots,n_x;\ j = 0,1,2\ldots,n_y;\ n = 0,1,2,\ldots\right)$; $\Delta x = 1/n_x$ is the mesh size in x-direction, $\Delta y = 1/n_y$ is the mesh size in y-direction, and Δt represents the time increment.

Newton's method is used to linearize the system of nonlinear Eqs. (40) and (41) and computed solution is obtained by iteration. The resulting linearized equations form a block tridiagonal matrix system of order n, as in the following form:

$$a_i\vec{\delta}_{i-1} + b_i\vec{\delta}_i + c_i\vec{\delta}_{i+1} = \vec{r}_i, \quad i = 1, 2, ..., n, \tag{43}$$

where a_i, b_i and c_i are block matrices of order two, $\vec{\delta} = [\delta u, \delta v]^T$ is the change in the solution vector, and $\vec{r}$ is the right hand-side vector, each of order two. At each iteration, the Gauss elimination method with partial pivoting algorithm is used to obtain the solution of the system (43). In the Iterative method, solution at the previous time step is taken as the initial guess for the convergence point of view. The convergence is obtained in one iteration.

The accuracy and consistency of the scheme is measured in terms of error norms L_1, L_2, L_∞ and relative error E_R, defined as:

$$\left.\begin{aligned}
L_1 &:= \| u_{exact} - u_{computed} \|_1 = \sum_{i=0}^{n}\sum_{j=0}^{n} | u_{i,j}^{\ exact} - u_{i,j}^{\ computed} | \\
L_2 &:= \| u_{exact} - u_{computed} \|_2 = \sqrt{\sum_{i=0}^{n}\sum_{j=0}^{n} | u_{i,j}^{\ exact} - u_{i,j}^{\ computed} |^2} \\
L_\infty &:= \| u_{exact} - u_{computed} \|_\infty = \max_{i,j} | u_{i,j}^{\ exact} - u_{i,j}^{\ computed} | \\
E_R &:= \left(\frac{\sum_{i=0}^{n}\sum_{j=0}^{n} | u_{i,j}^{\ exact} - u_{i,j}^{\ computed} |^2}{\sum_{i=0}^{n}\sum_{j=0}^{n} | u_{i,j}^{\ exact} |^2} \right)^{1/2}
\end{aligned}\right\}, \tag{44}$$

where u_{exact} and $u_{computed}$ represent exact and computed solutions, respectively.

4. Numerical results and discussions

We take $\mu = 1$ and $\kappa = \upsilon = \dfrac{1}{\mathrm{Re}}$ for the simple calculation purpose and discuss the following three cases on the uniform grid:

Case 1: $a_1 = 100$, $a_2 = 0$, $a_3 = 0$, $a_4 = 1$, $A = 1$, $B = 1$, $C = 1$, $D = 1$, $E = 0$, $\beta = \pi$ and $\gamma = \pi$, the analytical solutions of Eqs. (1) and (2) are given by

$$\left.\begin{aligned}
u &= \left(\frac{-2}{\mathrm{Re}}\right)\left(\frac{y+(\cos\pi x-\sin\pi x)\sin\pi y A e^{\left(\frac{-2\pi^2}{\mathrm{Re}}\right)t}}{100+xy+(\sin\pi x+\cos\pi x)\sin\pi y A e^{\left(\frac{-2\pi^2}{\mathrm{Re}}\right)t}}\right)\\
v &= \left(\frac{-2}{\mathrm{Re}}\right)\left(\frac{x+\pi(\sin\pi x+\cos\pi x)\cos\pi y A e^{\left(\frac{-2\pi^2}{\mathrm{Re}}\right)t}}{100+xy+(\sin\pi x+\cos\pi x)\sin\pi y A e^{\left(\frac{-2\pi^2}{\mathrm{Re}}\right)t}}\right)
\end{aligned}\right\}. \tag{45}$$

Case 2: $a_1 = 0,\ a_2 = 5,\ a_3 = 10,\ a_4 = 0,\ A = 1,\ B = 0,\ C = 1,\ D = 0,\ E = 1,\ \beta = 0$ and $\gamma = 2\pi$, the exact solutions of Eqs. (1) and (2) are given as

$$\left.\begin{aligned}
u &= \left(\frac{-2}{\mathrm{Re}}\right)\left(\frac{1}{x+2y+\frac{1}{5}\cos 2\pi y\, e^{\left(\frac{-4\pi^2}{\mathrm{Re}}\right)t}}\right)\\
v &= \left(\frac{-2}{\mathrm{Re}}\right)\left(\frac{10-2\pi\sin 2\pi y\, e^{\left(\frac{-4\pi^2}{\mathrm{Re}}\right)t}}{5x+10y+\cos 2\pi y\, e^{\left(\frac{-4\pi^2}{\mathrm{Re}}\right)t}}\right)
\end{aligned}\right\}. \tag{46}$$

Case 3: $a_1 = 10,\ a_2 = 50,\ a_3 = 0,\ a_4 = 0,\ A = 1,\ B = 0,\ C = 1,\ D = 1,\ E = 0,\ \beta = 2\pi$ and $\gamma = 2\pi$, the analytical solutions of Eqs. (1) and (2) are given by

$$\left.\begin{aligned}
u &= \left(\frac{-2}{\mathrm{Re}}\right)\left(\frac{50-2\pi\sin 2\pi x\sin 2\pi xy\, e^{\left(\frac{-8\pi^2}{\mathrm{Re}}\right)t}}{10+50x+\cos 2\pi x\sin 2\pi y\, e^{\left(\frac{-8\pi^2}{\mathrm{Re}}\right)t}}\right)\\
v &= \left(\frac{-2}{\mathrm{Re}}\right)\left(\frac{2\pi\cos 2\pi x\cos 2\pi y\, e^{\left(\frac{-8\pi^2}{\mathrm{Re}}\right)t}}{10+50x+\cos 2\pi x\sin 2\pi y\, e^{\left(\frac{-8\pi^2}{\mathrm{Re}}\right)t}}\right)
\end{aligned}\right\}. \tag{47}$$

Three sets of initial and boundary conditions are generated using above analytical conditions (45), (46) and (47) to obtain different numerical solutions. The effect on the computed solutions and the stability of the described scheme is investigated by varying Reynolds number and mesh size.

For Case 1, the computed and exact values of u and v are given in Tables 1 and 2 at some typical grid point (TGP). Tables 1 and 2 show that the described *Log* FDM achieves an excellent result with the exact solutions of the equations. Computed solutions of u and v are shown in Fig. 1 for $\mathrm{Re} = 500$, grid size 4×4, $\Delta t = 0.001$ at $t = 1.0$ using *Log* FDM.

Table 1. Comparison of computed and exact values of u for $\text{Re} = 500$, 20×20 grid and $\Delta t = 0.001$.

TGP	$t = 0.01$		$t = 0.5$		$t = 1.0$	
(x, y)	*Log FDM*	Exact	*Log FDM*	Exact	*Log FDM*	Exact
(0.1,0.1)	-2.881E-05	-2.881E-05	-2.834E-05	-2.833E-05	-2.786E-05	-2.786E-05
(0.5,0.1)	3.469E-05	3.469E-05	3.396E-05	3.395E-05	3.322E-05	3.321E-05
(0.9,0.1)	4.496E-05	4.496E-05	4.402E-05	4.402E-05	4.308E-05	4.308E-05
(0.1, 0.5)	-9.935E-05	-9.935E-05	-9.785E-05	-9.785E-05	-9.635E-05	-9.634E-05
(0.5, 0.5)	1.043E-04	1.043E-04	1.020E-04	1.020E-04	9.960E-05	9.959E-05
(0.9, 0.5)	1.386E-04	1.386E-04	1.355E-04	1.355E-04	1.325e-04	1.324E-04
(0.1, 0.9)	-6.063E-05	-6.063E-05	-6.016E-05	-6.016E-05	-5.969E-05	-5.969E-05
(0.5, 0.9)	2.796E-06	2.796E-06	2.059E-06	2.058E-06	1.322E-06	1.319E-06
(0.9, 0.9)	1.284E-05	1.283E-05	1.190E-05	1.190E-05	1.097E-05	1.097E-05

For Case 2, analytical solutions of u and v are shown in Fig. 2 and numerical solutions of u and v, are shown in Fig. 3 for $\text{Re} = 5000$, grid size 32×32, $\Delta t = 0.001$ at $t = 1.0$. From Fig. 2 and 3, it can be seen that numerical solutions are in excellent agreement with the corresponding analytical solutions.

Table 2. Comparison of computed and exact values of v for $\text{Re} = 500$, 20×20 grid and $\Delta t = 0.001$.

TGP	$t = 0.01$		$t = 0.5$		$t = 1.0$	
(x, y)	*Log FDM*	Exact	*Log FDM*	Exact	*Log FDM*	Exact
(0.1,0.1)	-1.539E-04	-1.539E-04	-1.511E-04	-1.511E-04	-1.482E-04	-1.482E-04
(0.5,0.1)	-1.390E-04	-1.390E-04	-1.367E-04	-1.367E-04	-1.344E-04	-1.344E-04
(0.9,0.1)	4.075E-05	4.075E-05	3.928E-05	3.927E-05	3.781E-05	3.780E-05
(0.1, 0.5)	-3.948E-06	-3.948E-06	-3.949E-06	-3.949E-06	-3.950E-06	-3.950E-06
(0.5, 0.5)	-1.975E-05	-1.975E-05	-1.976E-05	-1.976E-05	-1.976E-05	-1.976E-05
(0.9, 0.5)	-3.607E-05	-3.607E-05	-3.607E-05	-3.607E-05	-3.606E-05	-3.606E-05
(0.1, 0.9)	1.458E-04	1.458E-04	1.430E-04	1.430E-04	1.401E-04	1.401E-04
(0.5, 0.9)	9.872E-05	9.872E-05	9.646E-05	9.645E-05	9.419E-05	9.418E-05
(0.9, 0.9)	-1.120E-04	-1.120E-04	-1.106E-04	-1.106E-04	-1.091E-04	-1.091E-04

Table 3. Error norms L_1, L_2, L_∞ and E_R for the u-component at $\text{Re} = 50000$, 32×32 grid, $\Delta t = 0.001$ at $t = 1.0$.

Grid	L_1	L_2	L_∞	E_R
4×4	1.7298E-10	7.5147E-10	1.7191E-10	7.1910E-06
8×8	3.8512E-11	2.1443E-10	2.8908E-11	2.2945E-06
16×16	8.8915E-12	7.3942E-11	5.2728E-12	8.4347E-07
32×32	2.5834E-12	2.4132E-11	7.9248E-13	2.8450E-07

Table 4. Error norms L_1, L_2, L_∞ and E_R for the v-component at $\mathrm{Re} = 50000$, 32×32 grid $\Delta t = 0.001$ at $t = 1.0$.

Grid	L_1	L_2	L_∞	E_R
4×4	8.5826E-11	4.2899E-10	8.5830E-11	4.6856E-05
8×8	4.0690e-11	2.3693E-10	3.1231E-11	3.1980E-05
16×16	1.1268E-11	8.4451E-11	5.5482E-12	1.2995E-05
32×32	1.9372E-12	2.4809e-11	8.0799E-13	4.1068E-06

In the Case 3, exact solutions of u and v are shown in Fig. 4 and computed solutions of u and v, are shown in Fig. 5 for $\mathrm{Re} = 50000$, grid size 32×32, $\Delta t = 0.001$ at $t = 1.0$. Tables 3 and 4 show error norms L_1, L_2, L_∞ and E_R of u and v velocity components respectively, at $\mathrm{Re} = 50000$, grid size 32×32, $\Delta t = 0.001$, $t = 1.0$. From Tables 3 and 4, it is seen that errors approach zero as the mesh refines, this shows that the described scheme is consistent.

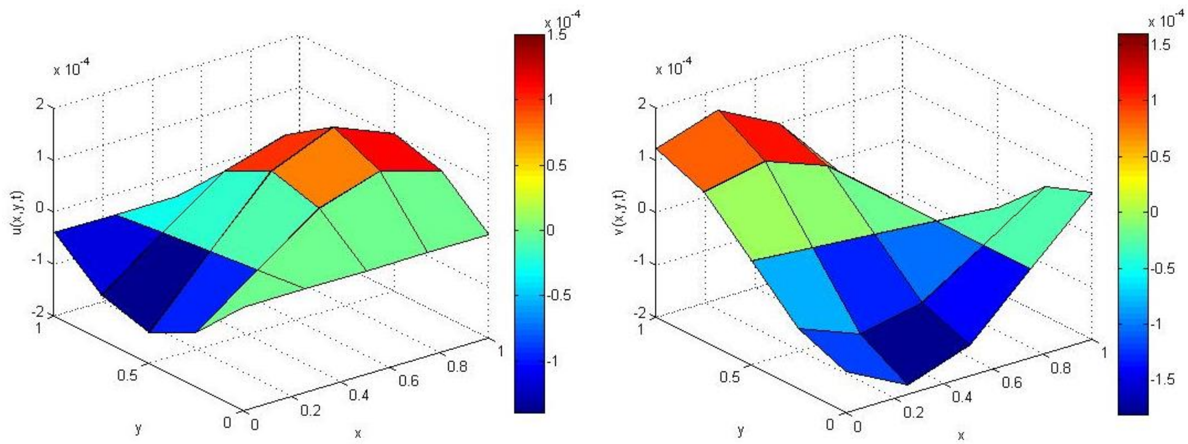

Fig. 1. Computed solution u (left) and v (right) for $\mathrm{Re} = 500$, 4×4 grid, $\Delta t = 0.001$ at $t = 1.0$ using *Log* FDM.

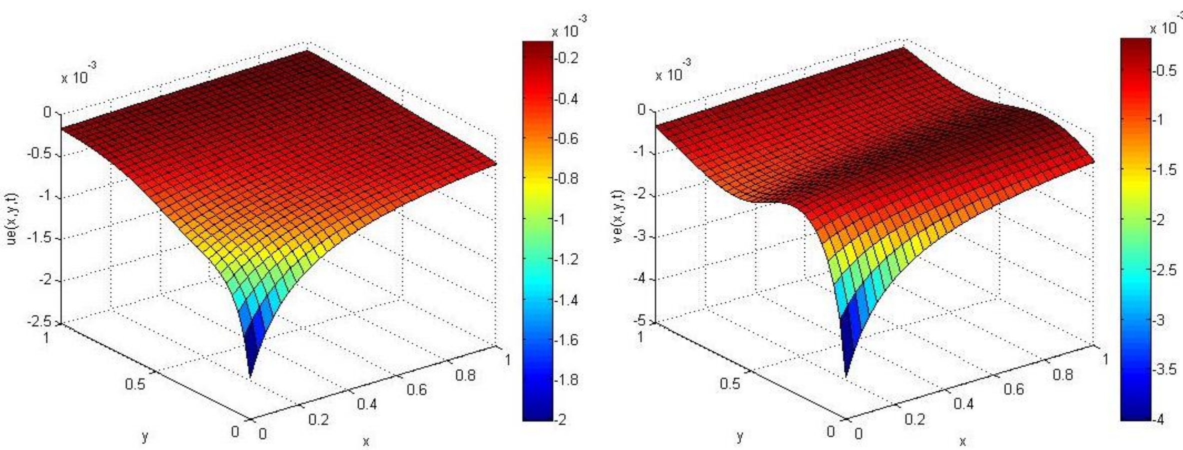

Fig. 2. Analytical solution u (left) and v (right) for $\mathrm{Re} = 5000$, 32×32 grid, $\Delta t = 0.001$ at $t = 1.0$

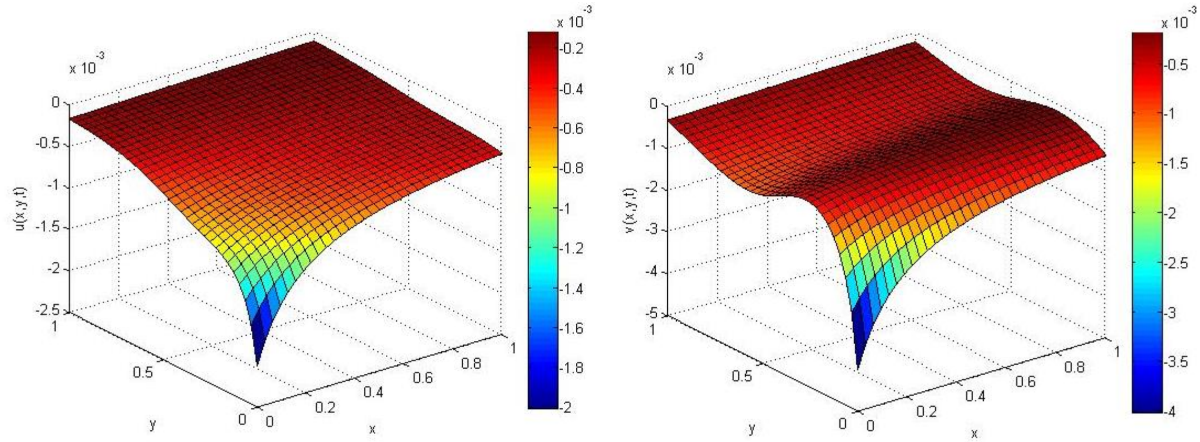

Fig. 3. Numerical solution u (left) and v (right) for $\mathrm{Re} = 5000$, 32×32 grid, $\Delta t = 0.001$ at $t = 1.0$ using Log FDM.

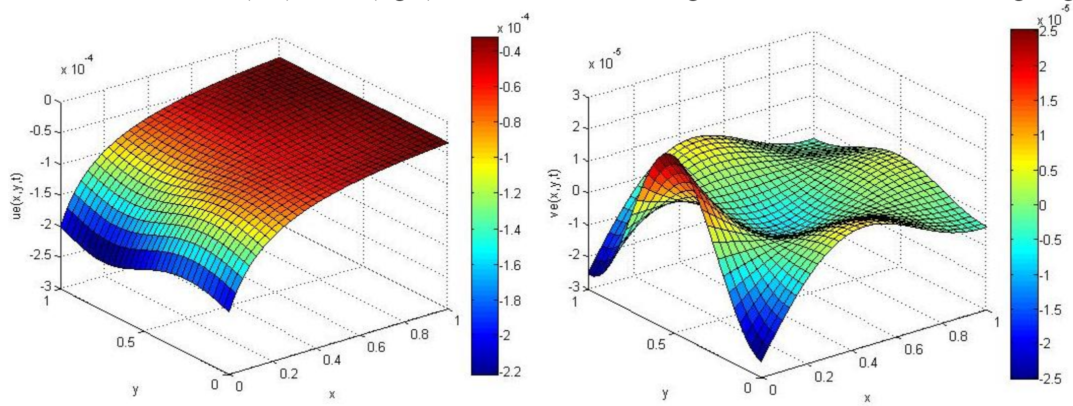

Fig. 4. Exact solution u (left) and v (right) for $\mathrm{Re} = 50000$, 32×32 grid, $\Delta t = 0.001$ at $t = 1.0$.

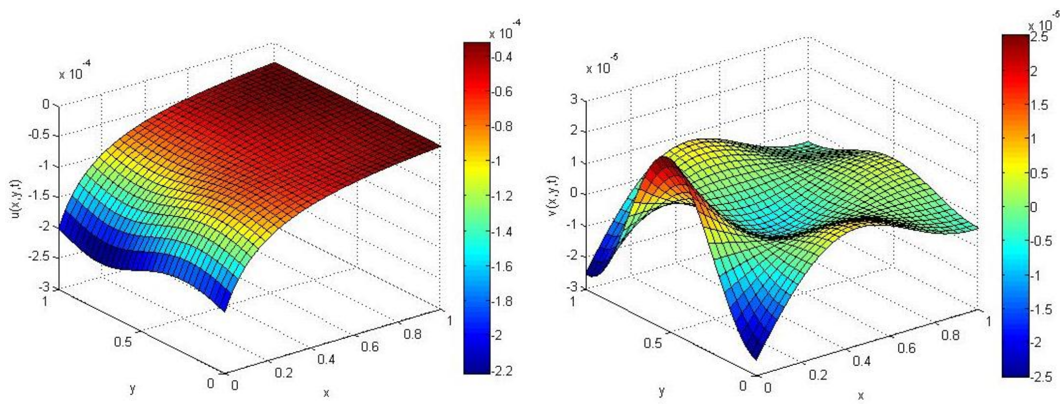

Fig. 5. Computed solution u (left) and v (right) for $\mathrm{Re} = 50000$, 32×32 grid, $\Delta t = 0.001$ at $t = 1.0$ using Log FDM.

5. Conclusions

A general analytical solution of two dimensional unsteady coupled Burgers' equations is derived with the help of Hopf-Cole transformation and separation of variables. The exact solution is used to generate three different sets of initial and boundary conditions, which are then used for numerical approximations of the Burgers' equations using the described implicit Logarithmic finite-difference method (*Log FDM*). Since the error norms approach zero as the mesh is refined, consistency of the scheme is achieved. Variation in the Reynolds number does not adversely affect the computed solutions.

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