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Stability analysis and internal heating effects on oscillatory convection in a viscoelastic fluid layer under gravity modulation

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ABSTRACT

In this paper, we study the combined effect of internal heating and time periodic gravity modulation on thermal instability in a viscoelastic fluid layer, using complex non-autonomous Ginzburg-Landau equation. The influence of relaxation and retardation time of viscoelastic fluid on heat transfer has been discussed. A weak non-linear stability analysis has been performed by using power series expansion in terms of the amplitude of gravity modulation, which is assumed to be small. The Nusselt number has been obtained in terms of the amplitude for oscillatory mode of convection. It is found that modulation has a destabilizing effect at low frequencies and stabilizing effect at high frequencies. Further, it is also found that overstability advances the onset of convection more with internal heating, hence increases heat transfer. We have also studied the subcritical Hopf bifurcation and pitchfork bifurcation. The phase portrait diagrams are also shown for these bifurcations.

Keywords: Non-linear stability analysis; Complex Ginzburg-Landau equation; Gravity modulation; Internal heating; Bifurcation

1. Introduction

Thermal convection in a horizontal fluid layer subject to constant but different temperatures at the boundaries has been studied extensively due to its applications in various fields. The classical Rayleigh-Benard convection is found to be an interesting phenomenon introduced by Chandrasekhar [1], due to bottom heating of a fluid layer. For analysis on thermal instability in detail, one may refer excellent books due to Ingham and Pop [2], Nield and Bejan [3] and vafai

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[4]. From the applications point of view, the regulation of convection is very important. Thermo-gravitational vibration is known to be an effective technique for controlling the instability. The first idea of using mechanical vibration as a tool to enhance the heat transfer rate is presented by Gresho and Saini [5]. Further studies on gravity modulation are due to Malashetty and Padmavathi [6], Malashetty and Swamy [7], Bhadauria et al. [8] and Siddhavaram and Hosmy [9].

Basically, industrial fluids are non-Newtonian fluids. Specially, viscoelastic fluids have been found a wide range of industrial applications. The characteristics of heat transfer in viscoelastic fluid layer are also important in chemical processing industries, and so, the proper understanding of convective motion and its behaviour is necessary for controlling many processes such as geothermal reservoirs, filtration, enhanced oil recovery etc. In the recent years, several articles are available in which different physical models with viscoelastic fluid layer have been used to study thermal instability. First of all Green [10] study the oscillatory convection in a viscoelastic fluid layer, the occurrence of over stability for typical Rayleigh-Benard convection in a horizontal layer of homogeneous Maxwellian fluid, heated from below, is reported by Vest and Arpaci [11]. Convective instability in a rotating viscoelastic fluid layer was studied by Bhatia and Steiner [12]. Thermal instability in a viscoelastic fluid saturating a porous medium was studied by Kim et al. [13]. The Benard-Marangoni thermal instability problem in a viscoelastic Jeffrey's fluid layer with internal heat generation was introduced by Comissiong et al. [14]; onset of oscillatory convection was studied using linear stability analysis and the dependence of critical Rayleigh number for oscillatory convection on internal heat generation, relaxation and retardation times was derived. Thermal instability using linear stability analysis was studied by Rajib and Layek [15]. Recently, Bhadauria and Kiran [16] investigated oscillatory convection in a viscoelastic fluid layer under gravity modulation by making a non-linear stability analysis.

Internal heat is one of the main sources of energy for celestial bodies, caused by nuclear fusion and decaying of radioactive materials. It is due to the internal heating of the earth that there exists a thermal gradient between the interior and exterior of the earth's crust, saturated by multi-components fluids, which helps convective flow, thereby transferring the thermal energy toward the surface of the earth, and so internal heat generation plays a very significant role in several applications which include geophysics, reactor safety analyses, metal waste form development for spent nuclear fuel, fire and combustion studies, and storage of radioactive materials. Convective heat transfer in porous media has received much attention during the past few decades because of its wide range of applications in many engineering and natural systems of practical interest; for instance, geothermal energy utilization, thermal energy storage and recovery systems, petroleum reservoirs, industrial and agricultural water distribution. Haajizadeh et al. [17] have studied a uniform heat generation term across an enclosure with isothermal vertical walls and adiabatic horizontal walls. Further, for more studies on internal heating we refer to [18- 26].

The qualitative analysis of a dynamical system provides much knowledge about the system. Bifurcations, the appearance of a topologically non-equivalent phase portrait under variation of parameters, are scientifically more important as they provide models of transitions and instabilities [27]. In the present paper, two types of bifurcation (1) Pitchfork bifurcation (common in physical problems that have a symmetry); (2) Hopf bifurcation (bifurcation corresponding to the presence of purely complex eigenvalue) are discussed, for details we refer to [28].

In the existing literature, no work is available on internal heating system for oscillatory mode of convection with viscoelastic fluid layer and bifurcation analysis of the system. Therefore, in this paper, we study internal heating effects for a weak non-linear oscillatory convection in a viscoelastic fluid layer under gravity modulation. The heat transfer rate is examined by computing the Nusselt number in terms of the amplitude of convection by solving the complex Ginzburg-Landau equation. We also study the Hopf bifurcation and Pitchfork bifurcation on different parameters, which gives the more control parameters to study the stability of the system.

2. Mathematical Structure

We consider an infinitely extended horizontal viscoelastic fluid layer of depth d, confined between two parallel planes, the lower plane at $z=0$ while upper one is at $z=d$. A Cartesian frame of reference is adopted in such a way that the origin lies on the lower plane and z axis is vertically upward. The fluid layer is heated from below and cooled from the above. The physical configuration of the model is depicted in Fig. 1. The Oberbeck Boussinesq approximation is adopted to solve the model equations. The governing equations of the flow and temperature fields [16] are given by

$$\begin{cases} \nabla.\vec{q}=0, \\ \left(\bar{\lambda}_1\dfrac{\partial}{\partial t}+1\right)\left(\dfrac{\partial\vec{q}}{\partial t}+(\vec{q}.\nabla)\vec{q}-\dfrac{1}{\rho_0}\nabla p+\dfrac{\rho\vec{g}}{\rho_0}\right)-\nu\left(\bar{\lambda}_2\dfrac{\partial}{\partial t}+1\right)\nabla^2\vec{q}=0, \\ \dfrac{\partial T}{\partial t}+(\vec{q}.\nabla)\mathrm{T}=\kappa_T\nabla^2 T+Q(\mathrm{T}-\mathrm{T}_0), \\ \rho=\rho_0\{1-\alpha_T(\mathrm{T}-\mathrm{T}_0)\}, \end{cases} \tag{1}$$

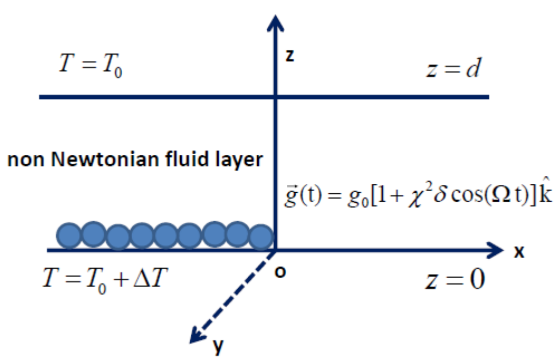


Fig.1. Physical configuration of the problem

where the physical meaning of the variables is as given in the nomenclature. The externally imposed gravitational field and the thermal boundary conditions are as follows

$$\vec{g} = g_0\{1+\chi^2\delta\cos(\Omega\,\mathrm{t})\}\hat{\mathrm{k}}, \tag{2}$$

$$\begin{cases} T = T_0 + \Delta T, & \text{at } \mathrm{z}=0, \\ \mathrm{T}_0, & \text{at } \mathrm{z}=\mathrm{d}, \end{cases} \tag{3}$$

where g_0 is the mean gravity and $\hat{k}$ is the unit vector along the positive z axis.

3. Basic state

The basic state is assumed to be quiescent, the quantities are taken as

$$\vec{q}_b = 0, \quad \mathrm{p}=\mathrm{p}_b(\mathrm{z}), \quad \mathrm{T}=\mathrm{T}_b(\mathrm{z}), \quad \rho=\rho_b(\mathrm{z}). \tag{4}$$

The following relations, which define basic state pressure and temperature mathematically, are obtained by putting Eq. (4) in Eq. (1):

$$\frac{\partial p_b}{\partial z} = -\rho_b\,\vec{\mathrm{g}}, \tag{5}$$

$$\kappa_T\frac{d^2(\mathrm{T}_b-\mathrm{T}_0)}{dz^2} + Q(\mathrm{T}_b-\mathrm{T}_0) = 0, \tag{6}$$

$$\rho_b = \rho_0\{1-\alpha_T(\mathrm{T}_b-\mathrm{T}_0)\}. \tag{7}$$

The exact solution of Eq. (6) together with the boundary conditions (3) is given by

$$T_b = T_0 + \Delta T\frac{\sin\left(\left(\sqrt{\frac{Q}{\kappa_T}}\right)\left(1-\frac{z}{d}\right)\right)}{\sin\left(\sqrt{\frac{Q}{\kappa_T}}\right)}. \tag{8}$$

Now, we superimpose the finite amplitude perturbations on the basic state in the form:

$$\vec{q} = q_b + q', \quad \mathrm{T}=\mathrm{T}_b + T', \quad \mathrm{p}=\mathrm{p}_b + p', \quad \rho=\rho_b + \rho', \tag{9}$$

the primes denote the perturbed quantities. The dimensionless governing system as mentioned in [16] is reduces to

$$\left(\lambda_1\frac{\partial}{\partial t}+1\right)\left(\frac{1}{P_r}\frac{\partial}{\partial t}\nabla^2\psi - \frac{1}{P_r}\frac{\partial(\psi,\nabla^2\psi)}{\partial(\mathrm{x},\mathrm{z})} + g_m R_a\frac{\partial T}{\partial x}\right) - \left(\lambda_2\frac{\partial}{\partial t}+1\right)\nabla^4\psi = 0, \tag{10}$$

$$-\frac{\partial\psi}{\partial x}\frac{\partial T_b}{\partial z} + \left(\frac{\partial}{\partial t} - \nabla^2 - \mathrm{R}_i\right)T = \frac{\partial(\psi,T)}{\partial(x,z)}, \tag{11}$$

where $g_m = (1+\chi^2\delta\cos(\Omega t))$, $R_a = \frac{\alpha_T g_0\Delta T K d^3}{\nu\kappa_T}$ is the thermal Rayleigh number, $R_i = \frac{Qd^2}{\kappa_T}$ is the internal Rayleigh number, $\nu = \frac{\mu}{\rho_0}$ is the kinematic viscosity and $P_r = \frac{\nu}{\kappa_T}$ is the Prandtl number. The above system will be solved by considering stress free and isothermal boundary conditions:

$$\psi=\frac{\partial^2\psi}{\partial z^2}=T=0 \;\text{ on }\; z=0,\, z=1. \tag{12}$$

The dimensionless steady temperature $T_b(z)$, appearing in Eq. (11) is

$$\frac{dT_b}{dz}=-\frac{\sqrt{R_i}\cos\left(\sqrt{R_i}(1-z)\right)}{\sin\left(\sqrt{R_i}\right)}. \tag{13}$$

On introducing a small perturbation parameter χ: a deviation from the critical state of onset of convection, the variables for a weak non-linear state may be expanded in power series of χ [29, 30] a

$$R_a=R_0+\chi^2R_2+\chi^4R_4+\cdots \tag{14}$$

$$\psi=\chi\psi_1+\chi^2\psi_2+\chi^3\psi_3+.... \tag{15}$$

$$T=\chi T_1+\chi^2T_2+\chi^3T_3+.... \tag{16}$$

where R_0 denotes the critical value of the Rayleigh number for the onset of convection in the absence of gravity modulation.

4. Analysis of the periodic solutions

In order to study the time periodic convective phenomenon, the slow and fast time scales are introduced $\left(\frac{\partial}{\partial t}=\frac{\partial}{\partial\tau}+\chi^2\frac{\partial}{\partial s}\right)$ by [13, 16]. The above system (10) and (11) is solved for each order of χ.

For the first order, the matrix operator is obtained similar to linear case as:

$$\begin{bmatrix} \frac{1}{P_r}\left(\lambda_1\frac{\partial}{\partial\tau}+1\right)\frac{\partial}{\partial\tau}\nabla^2-\left(\lambda_2\frac{\partial}{\partial\tau}+1\right)\nabla^4 & \mathrm{R}_0\left(\lambda_1\frac{\partial}{\partial\tau}+1\right)\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial x}\frac{\partial T_b}{\partial z} & \left(\frac{\partial}{\partial\tau}-\nabla^2-R_i\right) \end{bmatrix}\begin{bmatrix}\psi_1\\ T_1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}. \tag{17}$$

The solution of the first order system subject to the boundary conditions Eq. (12), is assumed to be

$$\psi_1=(\mathrm{B(s)}\,\mathrm{e}^{i\omega\tau}+\bar{\mathrm{B}}(\mathrm{s})\,\mathrm{e}^{-i\omega\tau})\sin ax\,\sin\pi z, \tag{18}$$

$$T_1=(\mathrm{A(s)}\,\mathrm{e}^{i\omega\tau}+\bar{A}(\mathrm{s})\,\mathrm{e}^{-i\omega\tau})\cos ax\,\sin\pi z, \tag{19}$$

where the unknown amplitudes are functions of slow time scale, and are related by the following expression:

$$B(s) = -\frac{(c+i\omega - R_i)(4\pi^2 - R_i)}{4\pi^2 a} A(s), \quad c = a^2 + \pi^2. \tag{20}$$

The values of the critical Rayleigh number and the corresponding wave number of the system for a stationary mode of convection are as given below:

$$R_0^{st} = \frac{c^2(c - R_i)(4\pi^2 - R_i)}{4\pi^2 a^2}, \tag{21}$$

$$a_c^2 = \frac{(R_i - \pi^2) \pm \sqrt{(\pi^2 - R_i)^2 + 8\pi^2(\pi^2 - R_i)}}{4}. \tag{22}$$

In particular, for $R_i = 0$ (without internal-heating), we have

$$R_0 = \frac{c^3}{a^2}, \tag{23}$$

$$a_c = \frac{\pi}{\sqrt{2}}, \tag{24}$$

which are the classical results as obtained by Chandrasekhar [1]. The critical Rayleigh number for the oscillatory mode of convection is computed as follows

$$R_0^{osc} = \left(\frac{c^3}{a^2} + \frac{(\lambda_1 \omega^2 R_i c - P_r R_i c^2) - (\lambda_1 + \lambda_2 P_r)\omega^2 c(c+1)}{a^2 P_r}\right)\frac{4\pi^2 - R_i}{4\pi^2}, \tag{25}$$

where ω is the oscillatory frequency as given below

$$\omega^2 = \frac{-1 + \frac{R_i}{c} + (\lambda_1 - \lambda_2) P_r c - P_r(1 - \lambda_2 R_i + \lambda_1 R_i)}{\lambda_1(\lambda_1 + \lambda_2 P_r) - \lambda_1^2 \frac{R_i}{c}}, \tag{26}$$

The similar result is computed by Rajib and Layek [15] without internal-heating. We compute the wave number (i.e., the critical wave number for which the Rayleigh number is minimum). It is to be noted that the critical Rayleigh number and the corresponding wave number do not depend on relaxation (λ_1) and retardation (λ_2) time in stationary mode of convection but it is not so in case of oscillatory mode of convection. Since ω has to be positive and real, and so, from the relation (26), the necessary condition for oscillatory convection is obtained as

$$\lambda_1 > \lambda_2 + \frac{1 + P_r(1 + \lambda_1 R_i - \lambda_2 R_i) - R_i c}{c P_r}. \tag{27}$$

Now at second order, we have

$$\begin{bmatrix} \frac{1}{P_r}\left(\lambda_1 \frac{\partial}{\partial \tau} + 1\right)\frac{\partial}{\partial \tau}\nabla^2 - \left(\lambda_2 \frac{\partial}{\partial \tau} + 1\right)\nabla^4 & R_0\left(\lambda_1 \frac{\partial}{\partial \tau} + 1\right)\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial x}\frac{\partial T_b}{\partial z} & \left(\frac{\partial}{\partial \tau} - \nabla^2 - R_i\right) \end{bmatrix} \begin{bmatrix} \psi_2 \\ T_2 \end{bmatrix} = \begin{bmatrix} R_{21} \\ R_{22} \end{bmatrix}, \tag{28}$$

where

$$R_{21} = 0, \tag{29}$$

$$R_{22} = \frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x}. \tag{30}$$

The second order solution subject to the boundary condition (12) is given by

$$\psi_2 = 0, \tag{31}$$

$$\left(\frac{\partial}{\partial \tau} - \nabla^2 - R_i\right) T_2 = R_{22}. \tag{32}$$

Next, we compute the temperature fields having the frequency 2ω, and independent of fast time scale [13, 16]. Thus, second order temperature terms can be expressed in the following form:

$$T_2 = \{T_{20} + T_{22}\,\mathrm{e}^{2i\omega\tau} + \bar{T}_{22}\,\mathrm{e}^{-2i\omega\tau}\}\sin(2\pi\,\mathrm{z}), \tag{33}$$

where T_{22} and T_{20} are temperature fields having the terms with the frequency 2ω and independent of fast time scale, respectively. The solutions of the second order problems are

$$T_{20} = \frac{\pi a}{8\pi^2 - 2R_i}\{\mathrm{A(s)}\,\bar{\mathrm{B}}\mathrm{(s)} + \bar{\mathrm{A}}\mathrm{(s)}\,\mathrm{B(s)}\}, \tag{34}$$

and

$$T_{22} = \frac{\pi a}{8\pi^2 + 4i\omega - 2R_i}\mathrm{A(s)}\,\mathrm{B(s)}. \tag{35}$$

Horizontally averaged Nusselt number, $N_u(\mathrm{s})$ for the oscillatory mode of convection is given by

$$N_u(\mathrm{s}) = 1 + \left[\chi^2\left(\frac{\partial T_2}{\partial z}\right)_{z=0} \Big/ \left(\frac{\partial T_b}{\partial z}\right)_{z=0}\right]. \tag{36}$$

By using Eq. (13), (33), (34) and (35), we can simplify Eq. (36)

$$N_u(\mathrm{s}) = 1 + \left[\frac{(\mathrm{c}-\mathrm{R}_i)4\pi^2}{8\pi^2 - 2\mathrm{R}_i} + \frac{2\pi^2\sqrt{(\mathrm{c}-\mathrm{R}_i)^2 + \omega^2}}{\sqrt{(8\pi^2 - 2\mathrm{R}_i)^2 + 16\omega^2}}\right]\left(\frac{4\pi^2 - \mathrm{R}_i}{4\pi^2}\right)\frac{\tan\sqrt{\mathrm{R}_i}}{\sqrt{\mathrm{R}_i}}|A(\mathrm{s})|^2. \tag{37}$$

It is clear that the gravity modulation is effective at third order, and affects $N_u(\mathrm{s})$ through A(s), which is evaluated at third order. At the third order, we have

$$\begin{bmatrix} \frac{1}{P_r}\left(\lambda_1\frac{\partial}{\partial \tau}+1\right)\frac{\partial}{\partial \tau}\nabla^2 - \left(\lambda_2\frac{\partial}{\partial \tau}+1\right)\nabla^4 & \mathrm{R}_0\left(\lambda_1\frac{\partial}{\partial \tau}+1\right)\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial x}\frac{\partial T_b}{\partial z} & \left(\frac{\partial}{\partial \tau} - \nabla^2 - R_i\right) \end{bmatrix}\begin{bmatrix}\psi_3 \\ T_3\end{bmatrix} = \begin{bmatrix} R_{31} \\ R_{32}\end{bmatrix}, \tag{38}$$

where

$$R_{31}=\lambda_2\frac{\partial}{\partial s}\nabla^4\psi_1-R_0\lambda_1\frac{\partial}{\partial s}\frac{\partial T_1}{\partial x}-(R_2+R_0\delta\cos(\Omega s))\left(\lambda_1\frac{\partial}{\partial\tau}+1\right)\frac{\partial T_1}{\partial x}$$
$$-\frac{1}{P_r}\left(\lambda_1\frac{\partial}{\partial\tau}+1\right)\frac{\partial}{\partial s}\nabla^2\psi_1-\frac{1}{P_r}\lambda_1\frac{\partial}{\partial s}\left(\frac{\partial}{\partial\tau}\nabla^2\psi_1\right), \tag{39}$$

$$R_{32}=\frac{\partial\psi_1}{\partial x}\frac{\partial T_2}{\partial z}-\frac{\partial T_1}{\partial s}. \tag{40}$$

Using first and second order solutions, the expressions of R_{31} and R_{32} are determined. Now, under the solvability condition for the existence of third order solution, one may derive the complex Ginzburg-Landau equation for finite amplitude convection.

$$\frac{dA(\mathrm{s})}{ds}-\gamma^{-1}F(\mathrm{s})\,\mathrm{A(s)}+\gamma^{-1}k\left|A(\mathrm{s})\right|^2A(\mathrm{s})=0, \tag{41}$$

Where

$$\gamma=\left(1-a\Delta_1R_2\lambda_1+\frac{c^2\Delta_1\lambda_2(\mathrm{c+i}\,\omega-\mathrm{R}_i)(4\pi^2-\mathrm{R}_i)}{4\pi^2a}+\frac{c\Delta_1(\mathrm{c+i}\,\omega-\mathrm{R}_i)(1+2\lambda_1\mathrm{i}\,\omega)(4\pi^2-\mathrm{R}_i)}{4\pi^2aP_r}\right),$$

$$F(\mathrm{s})=\left\{a\Delta_1R_2(1+\mathrm{i}\,\omega\lambda_1)(1+\delta\cos(\Omega\mathrm{s}))\right\},\ \Delta_1=\left\{\frac{aP_r}{i\omega c(1+\mathrm{i}\,\omega\lambda_1)+(1+\mathrm{i}\,\omega\lambda_2)\,\mathrm{c}^2\,P_r}\right\}$$

$$k=\left\{\frac{(\mathrm{c-R}_i)(\mathrm{c+i}\,\omega-\mathrm{R}_i)(4\pi^2-\mathrm{R}_i)\pi^2}{16\pi^4}+\frac{\{(\mathrm{c-R}_i)^2+\omega^2\}(4\pi^2-\mathrm{R}_i)^2\pi^2}{(8\pi^2-2\,\mathrm{R}_i+4\,\mathrm{i}\,\omega)16\pi^4}\right\}.$$

Writing A(s) in the phase-amplitude form, we get

$$A(\mathrm{s})=\left|A(\mathrm{s})\right|e^{i\varphi}. \tag{42}$$

On substituting Eq. (42) in Eq. (41), we get the following expression for the amplitude $|A(\mathrm{s})|$ as

$$\frac{d\left|A(\mathrm{s})\right|^2}{ds}-2p_r\left|A(\mathrm{s})\right|^2+2l_r\left|A(\mathrm{s})\right|^4=0, \tag{43}$$

$$\frac{d(\mathrm{ph(A(s))})}{ds}=p_i-l_i\left|A(\mathrm{s})\right|^2, \tag{44}$$

where $\gamma^{-1}F(\mathrm{s})=p_r+ip_i,\ \gamma^{-1}k=l_r+il_i$ and ph(.) represents the phase shift. The Eq. (43) solved numerically using the function NDSolve of Mathematica, subject to the suitable initial condition $A(0)=\mathrm{a}_0$, where a_0 is the chosen initial amplitude of convection. In our computation, we assume $R_2=R_0$ to keep the parameters to a minimum.

5. Bifurcation Analysis

In this section we are interested in the study of dynamical behaviour of the Complex Ginzburg-Landau Eq. (41) and amplitude Eq. (43). We show that the Complex Ginzburg-Landau

Eq. (41) undergoes subcritical Hopf bifurcation whereas amplitude Eq. (43) undergoes pitchfork bifurcation.

5.1 Hopf bifurcation

The complex Ginzburg-Landau equation (41) can be written as

$$\frac{dx}{ds} = p_r x - p_i y - (l_r x - l_i y)(x^2 + y^2),$$
$$\frac{dy}{ds} = p_i x + p_r y - (l_i x + l_r y)(x^2 + y^2), \quad (45)$$

where

$$A(s) = x(s) + iy(s),\ p_r = R_e\left[\frac{F(s)}{\gamma}\right],\ p_i = I_m\left[\frac{F(s)}{\gamma}\right],\ l_r = R_e\left[\frac{k}{\gamma}\right],\ l_i = I_m\left[\frac{k}{\gamma}\right].$$

Clearly the origin is the equilibrium point of the system (45). The Jacobian matrix of the system (45) at the origin is

$$J = \begin{bmatrix} p_r & -p_i \\ p_i & p_r \end{bmatrix}.$$

The trace and determinant of the Jacobian matrix J is $tr(J) = 2p_r$ and $\det(J) = p_r^2 + p_i^2 > 0$ respectively.

If $p_r < 0$, both eigenvalues of the Jacobian matrix J have negative real parts, and hence the origin is asymptotically stable. If $p_r > 0$, then real parts of both eigenvalues of J are positive, which confirms that the origin is unstable. For $p_r = 0$, the eigenvalues of J are purely imaginary, and so, from implicit function theorem a Hopf bifurcation occurs and a periodic orbit arises as the stability of origin changes. Here, we assume λ_1 as bifurcation parameter, and sketch the phase portrait diagram for the system (45).

We consider $P_r = 1$, $R_i = 1$, $\lambda_2 = 0.1$, δ=0.3, and Ω=50. If $\lambda_1 = 0.4$, then $p_r < 0$, and from the above discussion, the origin is asymptotically stable. From Fig. 11, we can see that the origin is a stable focus, surrounded by an unstable unique limit cycle. If $\lambda_1 = 0.47$, then $p_r = 0$, and the origin is unstable focus, Fig 12. If $\lambda_1 = 0.5$, then $p_r > 0$, and from the above discussion, the origin is unstable. Fig. 13 confirms that the origin is a focus. Such type bifurcation is the subcritical Hopf bifurcation.

5.2 Pitchfork bifurcation

The system (43) can be written as

$$\frac{d|A(s)|}{ds} - p_r|A(s)| + l_r|A(s)|^3 = 0. \quad (46)$$

It is clear that the system (46) has three equilibrium points $|A(s)|=0$ for all value of p_r, l_r and $|A(s)| = \pm\sqrt{\frac{p_r}{l_r}}$ for $p_r > 0,\ l_r > 0,$ and the solution of the differential equation (46) is given by

$$|A(s)|^2 = \frac{A_0^2}{\frac{l_r}{p_r}A_0^2 + \left(1 - \frac{l_r}{p_r}A_0^2\right)e^{-2p_r s}},\ p_r > 0,\ l_r > 0, \tag{47}$$

where A_0 is the initial value of the amplitude. From Eq. (47), we see that solution trajectories approach $|A(s)|=0$ as $s \to -\infty$, grow towards $\sqrt{\frac{p_r}{l_r}}$ when $0 < A_0 < \sqrt{\frac{p_r}{l_r}}$ as $s \to \infty$, decrease towards $\sqrt{\frac{p_r}{lr}}$ when $A_0 > \sqrt{\frac{p_r}{l_r}}$ as $s \to \infty$. Thus, if $p_r < 0$ then $A(s) = 0$ is the only equilibrium point which is stable. If $P_r = 0$ then origin again the only equilibrium point, which is still stable but much more weakly. So, if $P_r > 0$ and $l_r > 0$ then $|A(s)| = 0$ is still an equilibrium point but now it becomes unstable, and two new stable equilibrium points appear on either side of $|A(s)| = 0,$ symmetrically located at $|A(s)| = \pm\sqrt{\frac{p_r}{l_r}}$, Fig. 14. This is called the supercritical pitchfork bifurcation. The pitchfork bifurcation diagram is depicted in Fig. 15.

6. Results and Discussion

In present paper, the combined effect of internal-heating and gravity modulation on oscillatory convection in a viscoelastic fluid layer has been studied by performing a weak non-linear stability analysis. The effect of gravity modulation on the Rayleigh-Benard system has been assumed to be of order of (χ^2). The values of δ are assumed to be in the interval $(0, 0.5)$. It is observed that Eq. (27) leads to an interesting result that the oscillatory type of instability exists only when the relaxation parameter λ_1 is greater than the retardation parameter λ_2. From Eq. (37), it can be seen that the value of N_u starts with 1, thus showing the conduction state initially, that is heat transfer across the fluid layer is taking place through conduction only when s is small. The value of N_u increases for intermediate values of s thus showing that convection is in progress, and finally when s is very large, the oscillatory state is achieved. The numerical value of N_u is obtained from Eq. (37) by solving the amplitude equation (43).

The combined effect of internal-heating and gravity modulation has been depicted in Figs. 2-8, where we have plotted N_u with respect to the slow time s. It is evident from Fig. 2 that the effect of internal Rayleigh number on thermal instability is destabilizing as N_u increases on

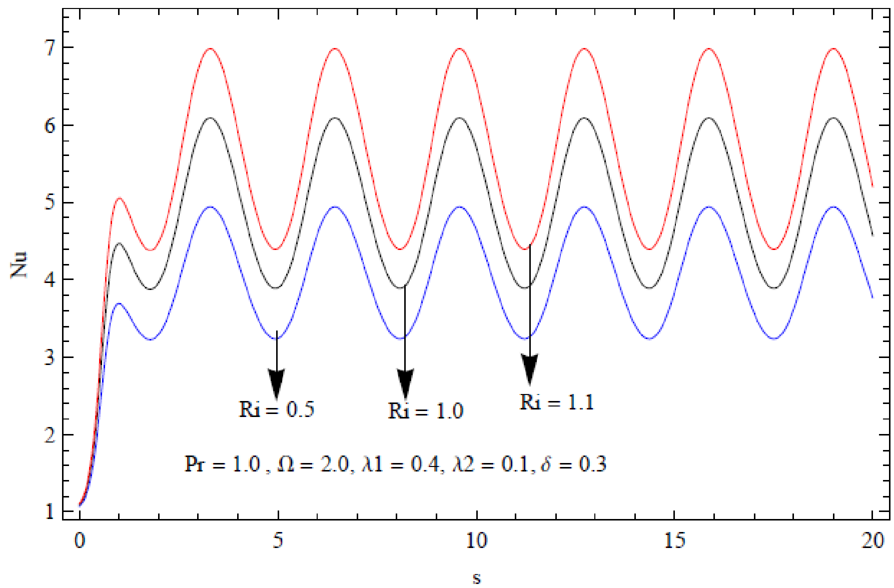


Fig .2. Effect of Ri on Nu For fixed values of the other parameters

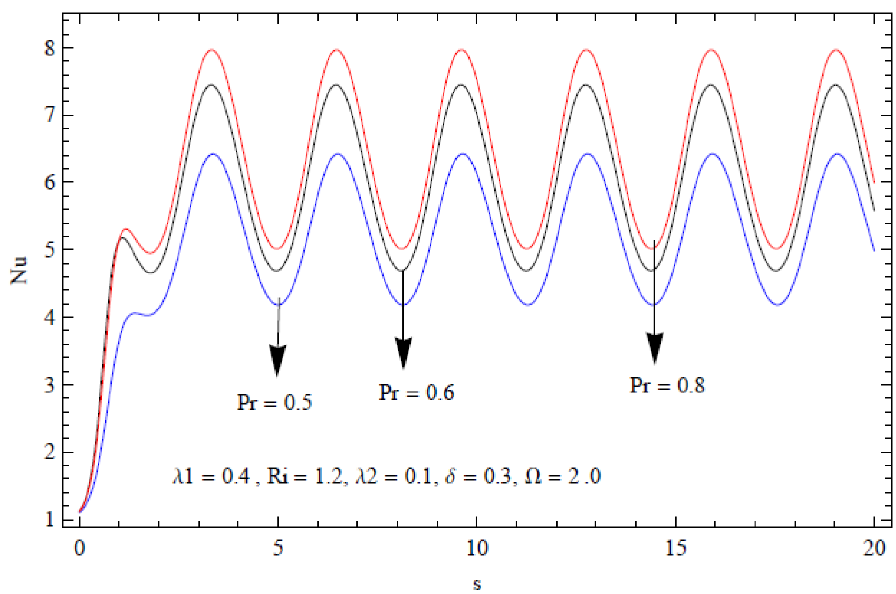


Fig .3. Effect of Pr on Nu For fixed values of the other parameters

increasing R_i, thus the heat transport is more for higher values of R_i. This agreed the results obtained by Bhadauria et al. [25]. Fig. 3 confirms that as P_r increases there is an increment in the heat transfer compatible with the results obtained by Bhadauria and kiran [16], thus the Prandtl number has a tendency to destabilize the system. Fig. 4 indicates the effect of relaxation parameter λ_1 on oscillatory convection, and gives a destabilized system due to increasing heat transfer on increasing λ_1. Further, the effect of retardation parameter λ_2 is found to stabilize the system as the heat transfer decreases on increasing λ_2, given in Fig. 5. The effects of the amplitude of modulation δ and frequency of modulation Ω on heat transport are given in Fig. 6-7 respectively. In Fig. 6, one can see that an increment in the amplitude of modulation increases the magnitude of N_u, thus enhances the heat transfer and advancing the onset of convection. An opposite effect is obtained in the case of frequency of modulation Ω as given in Fig. 7. Hence, we found that the effect of gravity modulation decreases as the frequency of modulation increases.

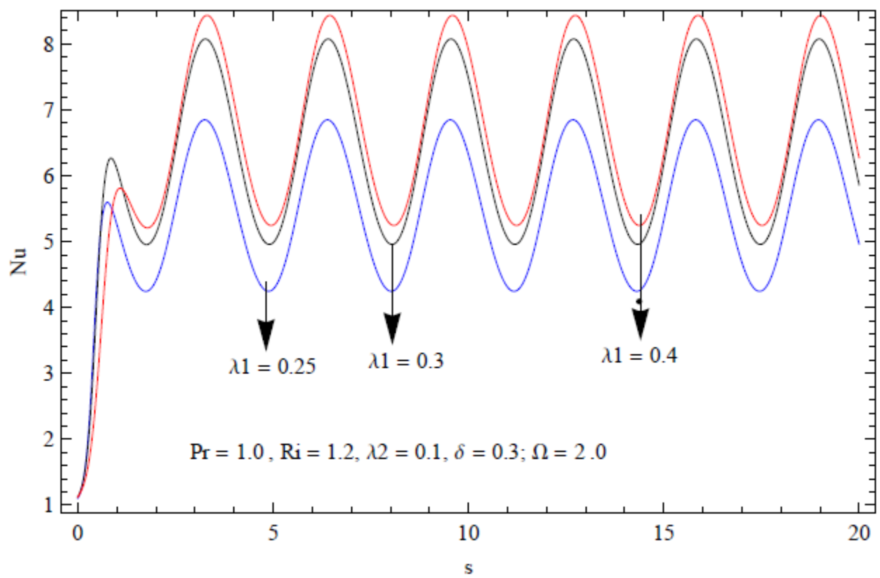


Fig .4. Effect of λ1 on Nu For fixed values of the other parameters

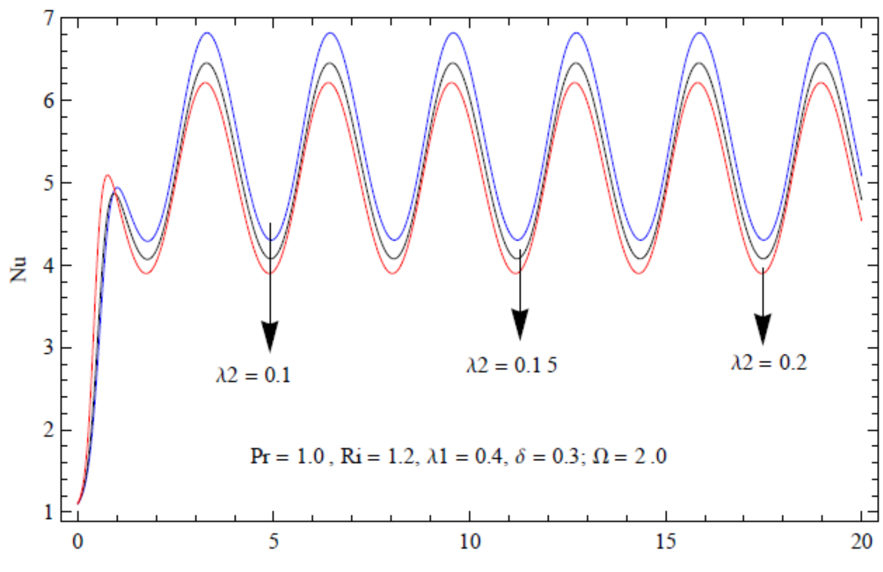


Fig .5. Effect of λ2 on Nu For fixed values of the other parameters

The present result of internal heating has been compared with the results of non-internal heating in Fig. 8. We observe that in the presence of internal heat source in the system, the value of N_u is more than that in the absence of internal-heating, i.e. the heat transport in the system is more due to internal-heating. Thus, internal-heating advances the onset of convection, which is the same as the result obtained by Bhadauria et al. [26].

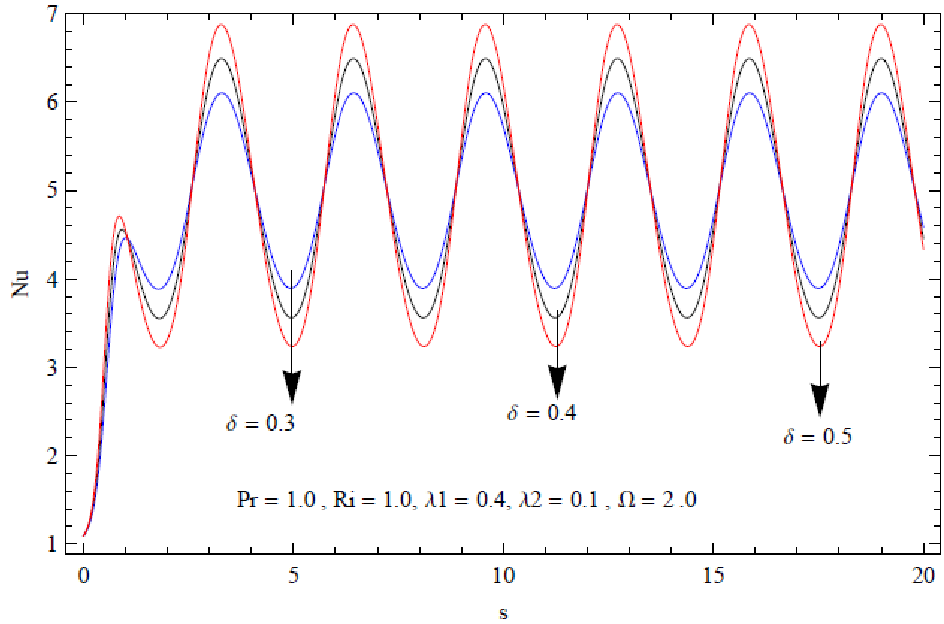


Fig .6. Effect of δ on Nu For fixed values of the other parameters

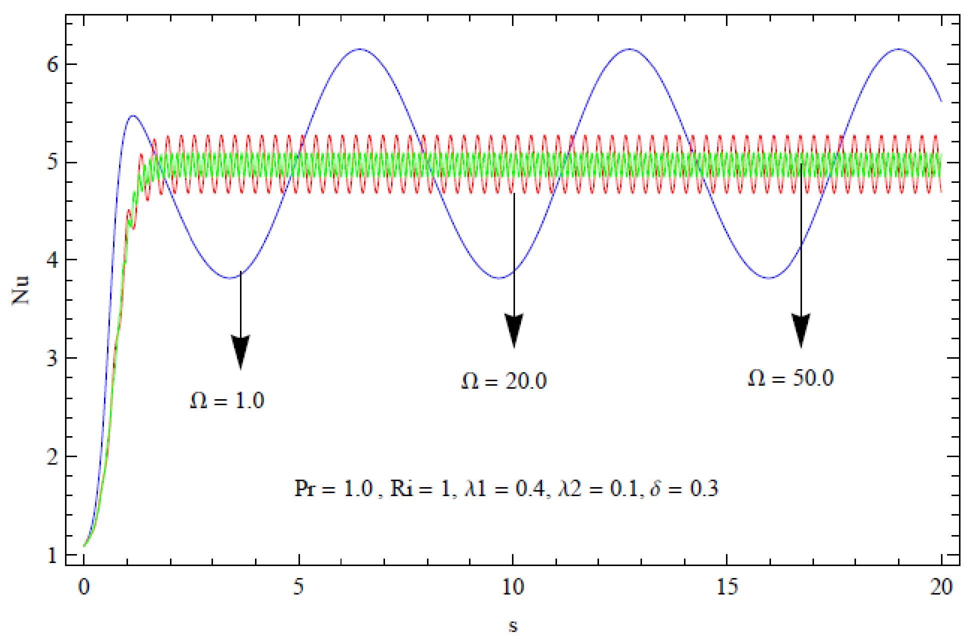


Fig .7. Effect of Ω on Nu For fixed values of the other parameters

In Fig. 9-10, the streamlines and corresponding isotherms are depicted respectively for different values of $s = 0.0,\ 0.4,\ 0.8,\ 1.0,\ 1.2,\ 1.5$ with $\lambda_1 = 0.4$, $\lambda_2 = 0.1$, $\delta = 0.1$, $\Omega = 2$, $P_r = 1$, $\chi = 0.5$ and $R_i = 0.1$. From these figures, we observed that initially when time is small, the magnitude of streamlines is also small, as given in Fig. 9 (a, b), and isotherms are straight, that is the system is in the conduction state, Fig. 10 (a, b). However, as time increases, the magnitude of streamlines increases and the isotherms lose their evenness Fig. 9 (c, d)-10 (c, d). This shows that convection is taking place in the system. The system achieves the steady state beyond $s = 1.0$ as there is no further change in the streamlines and isotherms Fig. 9 (e, f) - 10 (e, f).

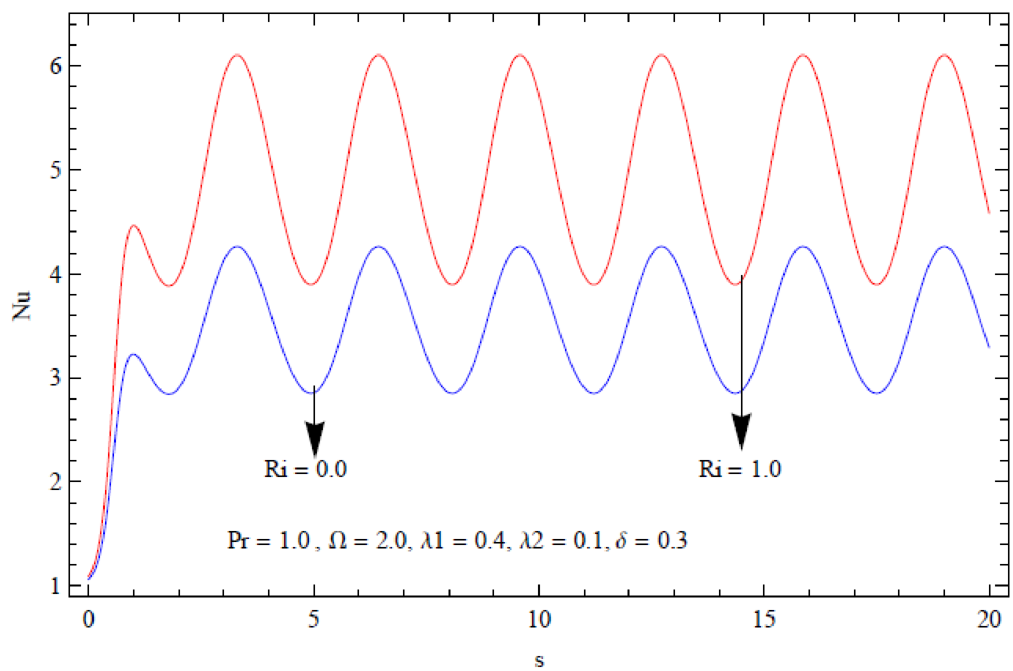


Fig .8. Comparison between internal and non internal - heating system

It is also shown that the system represented by Landau equation (41) enters subcritical Hopf bifurcation as stress relaxation time λ_1 is taken as the bifurcation parameter. Thus there exist critical value of λ_1 such that if λ_1 is less than the critical value then the system is stable and if λ_1 is greater than the critical value then the system is unstable. The phase portrait diagrams for the subcritical Hopf bifurcation are shown in Figs. 11-13. We have also discussed the pitchfork bifurcation for the amplitude equation (43) on the parameter amplitude of gravity modulation δ. The phase portrait diagram is shown in Fig. (14). The supercritical Pitchfork bifurcation diagram in Fig. (15) and subcritical Pitchfork bifurcation diagram in Fig. (16) show that the system becomes unstable as δ increases.

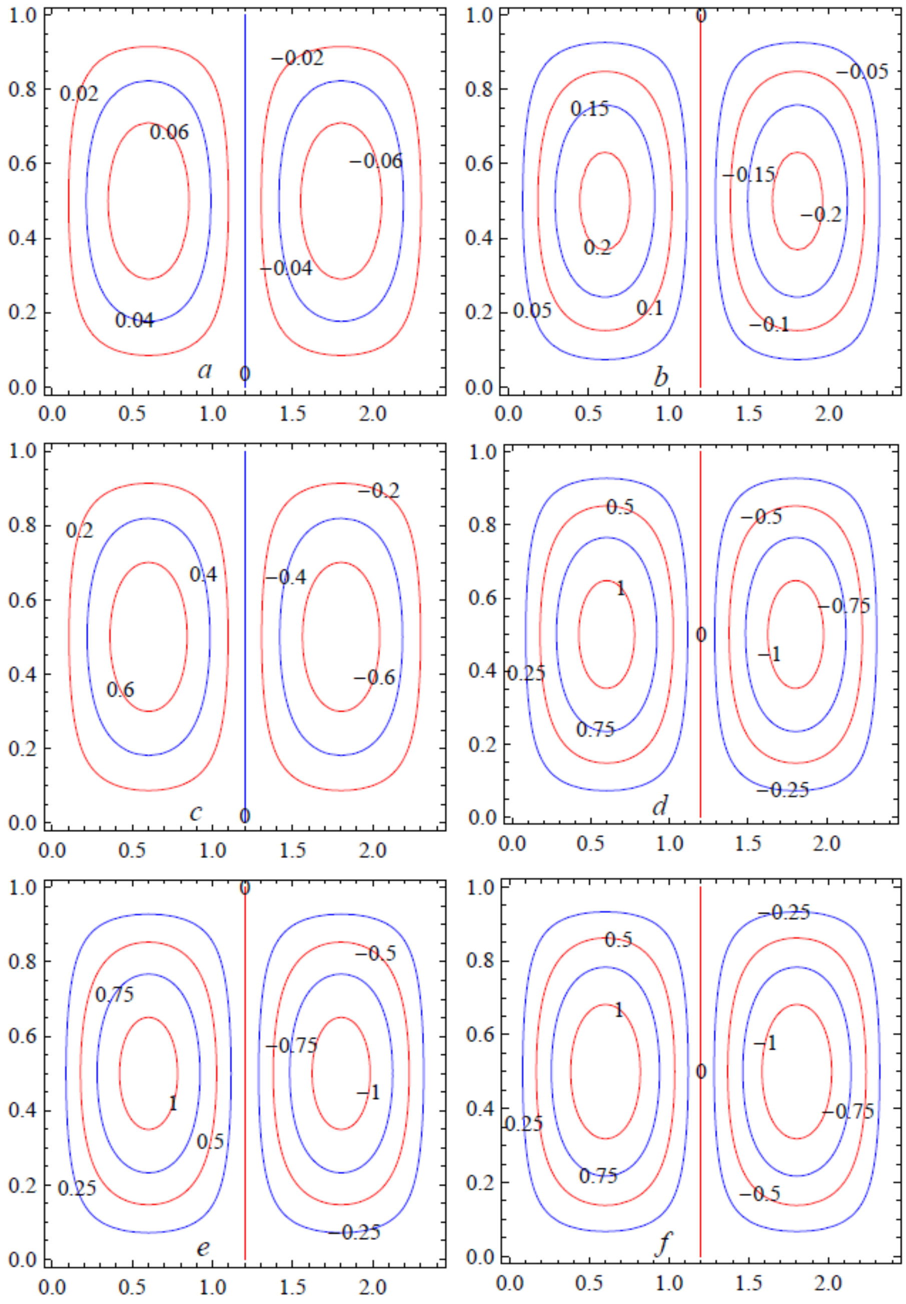


Fig .9 Stremlines at
(a) s = 0.0, (b) s = 0.4, (c) s = 0.8,
(d) s = 1.0, (e) s = 1.2, (f) s = 1.5

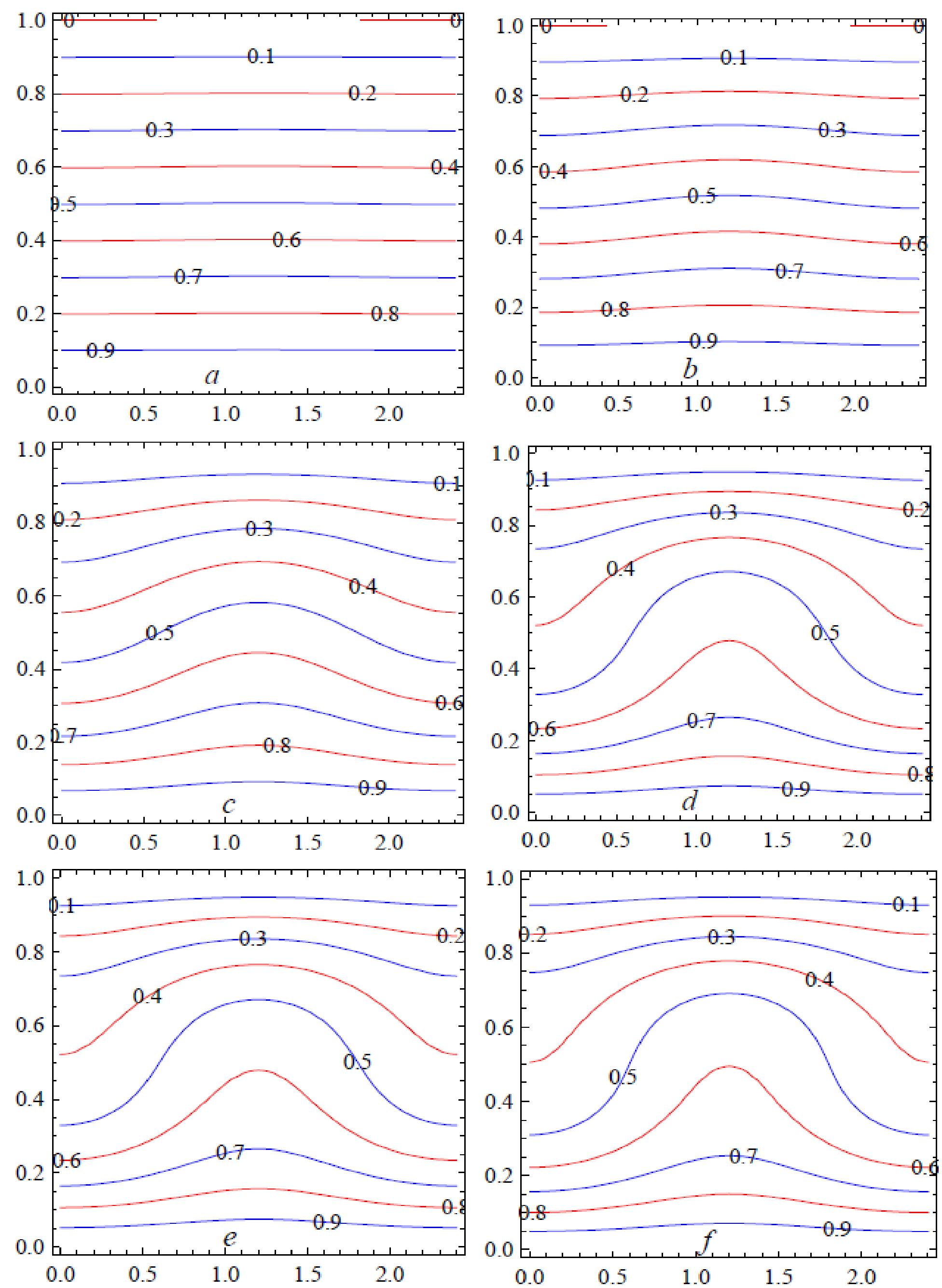
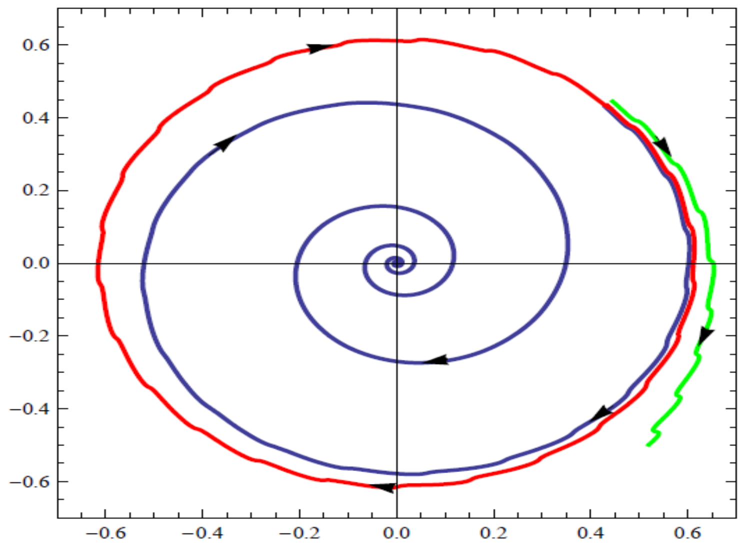
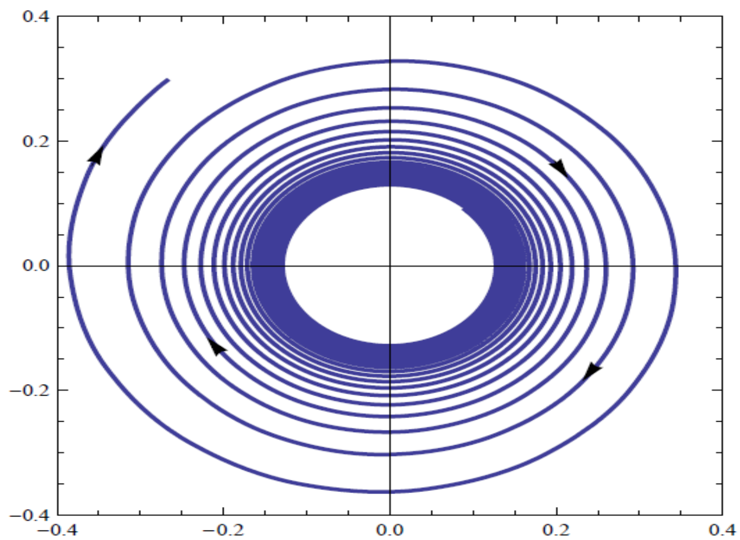


Fig .10 Isotherms at
(a) s = 0.0, (b) s = 0.4, (c) s = 0.8,
(d) s = 1.0, (e) s = 1.2, (f) s = 1.5



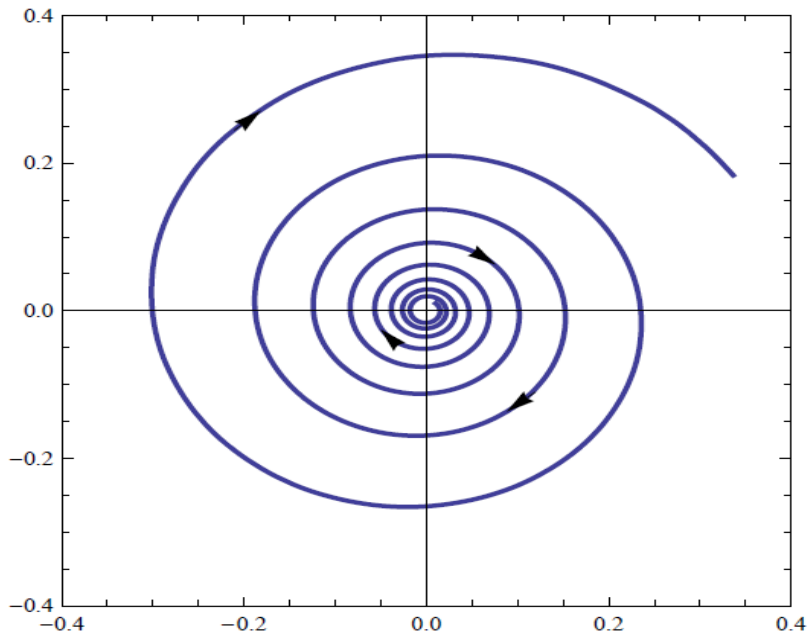
Pr = 1, Ri = 1, λ1 = 0.4, λ2 = 0.1, δ = 0.3, Ω = 50.

Fig. 11 An unstable limt cycle (red) is created thrugh Hopf bifurcation. The origin is asymptotically stable.



Pr = 1, Ri = 1, λ1 = 0.47, λ2 = 0.1, δ = 0.3, Ω = 50.

Fig. 12 Hopf bifurcation diagram.



Pr = 1, Ri = 1, λ1 = 0.5,
λ2 = 0.1, δ = 0.3, Ω = 50.

Fig .13 The origin is unstable.

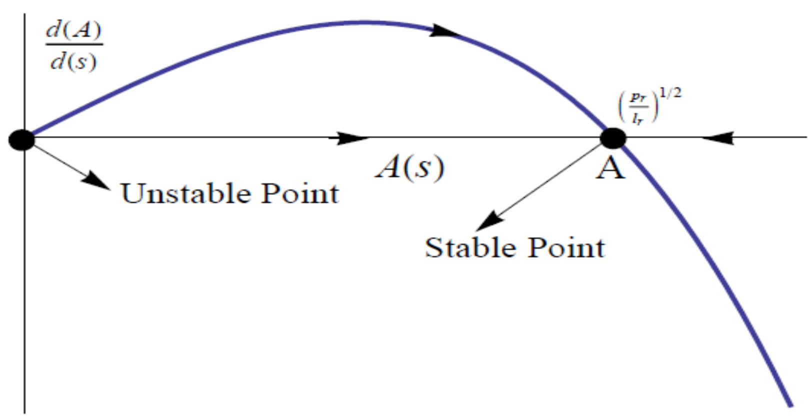
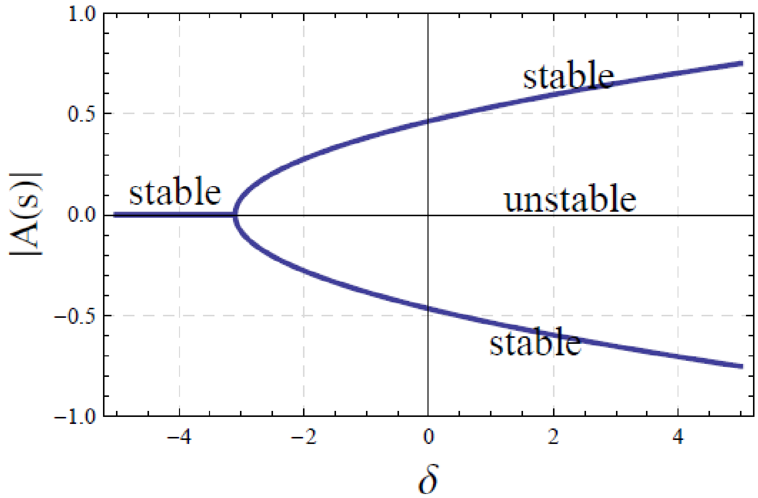
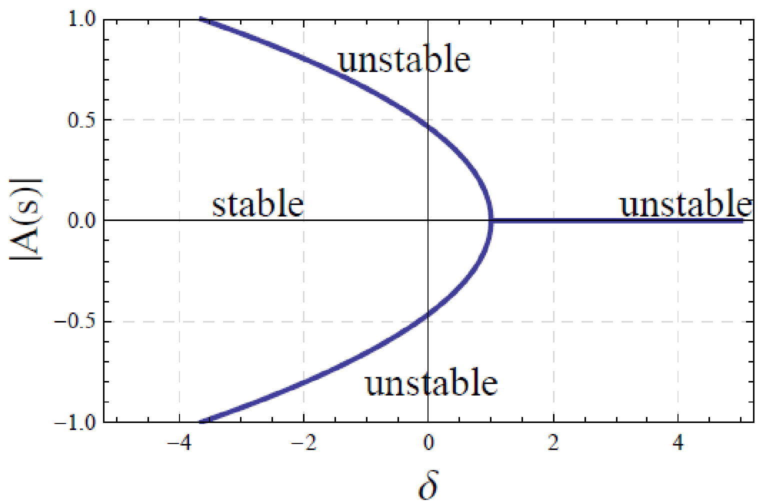


Fig. 14 Pr = 1, Ri = 1, λ1 = 0.4 , λ2 = .1, a = 2.61411, δ = 0.3, Ω = 1, s = 1000.



$P_r = 1, R_i = 1, \lambda_1 = 0.4, \lambda_2 = .1, a = 2.61411, \Omega = 1, s = 2000.$

Fig. 15 Supercritical Pitchfork Bifurcation.



$P_r = 1, R_i = 1, \lambda_1 = 0.4, \lambda_2 = 0.1, a = 2.61411, \Omega = 1, s = 2100.$

Fig .16 Subcritical Pitchfork bifurcation.

7. Conclusions

In the present paper, we consider the combined effect of internal heating and gravity modulation on oscillatory convection in a viscoelastic fluid layer, and perform a weak non-linear stability analysis by using the Ginzburg-Landau equation.

The following conclusions are drawn:

a) Heat transport is more in this case than in the absence of internal heating.

b) It is important that for the oscillatory convection the relaxation time λ_1 of fluid must be greater than the retardation time λ_2.

c) Effect of relaxation time λ_1 is to advance the onset of convection and hence enhances the heat transport.

d) Effect of retardation time λ_2 is to delay the onset of convection and hence decreases the heat transport.

e) An increment in the amplitude of modulation δ is to advance the onset of convection and thus increasing the heat transfer.

f) The frequency of modulation Ω is to decrease the heat transfer.

g) An increment in the value of prandtl number P_r destabilizes the system, thus heat transfer increases.

h) The system is stable if relaxation time λ_1 is less than the critical point and becomes destabilized if the relaxation time λ_1 is greater than the critical point.

i) The system is destabilized as the amplitude of gravity modulation δ increases in bifurcation analysis.

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Nomenclature

Nomenclature			
Latin symbols		μ	dynamic viscosity of fluid
A(s)	amplitude of convection	ν	kinematic viscosity
a	wave number	ρ	fluid density
δ	amplitude of gravity modulation	ψ	stream function
d	depth of fluid layer	s	slow time scale
g	acceleration due to gravity	χ	perturbation parameter
N_u	Nusselt number	τ	fast time scale
p	reduced pressure	Ω	frequency of modulation
R_a	thermal Rayleigh number	ω	oscillatory frequency
T	temperature		
ΔT	temperature difference	**Superscripts**	
t	time	,	perturbed quantity

$\vec{q}$	fluid velocity (u, v, w)	*	dimensionless quantity
Q	internal heat source	**Subscripts**	
R_i	internal Rayleigh number	b	basic state
(x, z)	horizontal and vertical co-ordinates	c	critical
Greek symbols		0	reference value
α_T	coefficient of thermal expansion		
K	permeability	**Other symbols**	
k_T	effective thermal diffusivity	$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$	
$\bar{\lambda}_1$	stress relaxation time		
$\bar{\lambda}_2$	strain retardation time		

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