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Indicatrix as a pair of planes and right circular cylinder for three dimensional Finsler Space

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ABSTRACT

In this paper, we consider two cases of three-dimensional Finsler Space, namely, indicatrix at any point as a pair of planes, and a right circular cylinder. We also use the notion of density at any point in the space. We then obtain fundamental function in each case and show that any geodesic in three-dimensional Finsler Space where indicatrix at any point is right circular cylinder is logarithmic spiral.

Keywords: Three dimensional Finsler Space; Fundamental function; Geodesics; Right circular cylinder

1. Introduction

Ingarden and Tamassy [6, 7] considered a Minkowski plane having a parabola as the indicatrix given by $y = x^2$ in [6] and $y = \left(\frac{k}{2}\right)x^2$ in [7], where k is an arbitrary constant. This structure is regarded as a Finsler Space with a special Kropina metric [2] while Matsumoto [1] considered two kinds of Finsler planes which are locally Ingarden-Tamassy's Minskowski plane. The metric of these Finsler planes give remarkable examples of 1-form metric [3] and of a Finsler Space having logarithmic spiral as geodesic.

In this paper, we have considered two cases of three-dimensional Finsler Space, first indicatrix at any point as a pair of planes and secondly it is a right circular cylinder. We also used the notion of density $\delta(x, y, z)$ at any point $P(x, y, z)$. We have obtained fundamental function $L(x, y, z, \dot{x}, \dot{y}, \dot{z})$ in each case and shown that any geodesics in three-dimensional Finsler Space where indicatrix at any point is right circular cylinder are logarithmic spiral. We also discuss the necessary and sufficient condition under which geodesics become logarithmic spiral in case of pair of planes as indicatrix at any arbitrary point.

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2. Indicatrix at any arbitrary point as pair of planes

Let us consider three dimensional Finsler Space $\mathbb{R}^3$ having an orthonormal coordinate system (x, y, z) with pole O and let $\delta(x, y, z)$ is a positive valued function defined on $\pi = \mathbb{R}^3 - \{0\}$. Then we defined a Finsler metric on π as follows:

For an arbitrary point $P(x, y, z)$ of π (refer Fig. 1) we take two points F_1 and F_2, and two planes P_1 and P_2 such that:

(i) PF_1 and PF_2 are orthogonal to OP and their Euclidean lengths are equal to $\delta(x, y, z)$-fold of OP.

(ii) P_1 and P_2 respectively are parallel to OP and through F_1 and F_2, respectively.

We choose F_1 and F_2 as

$F_1\left(x+\delta(x, y, z)x, y+\delta(x, y, z)y, z+\delta(x, y, z)z\right)$ and

$F_2\left(x-\delta(x, y, z)x, y-\delta(x, y, z)y, z-\delta(x, y, z)z\right)$, respectively.

Therefore, the equation of planes P_1 and P_2, in terms of current coordinate, (u, v, w), are expressed as:

$$P_1: \ \delta\left[x(u-x-\delta x)+y(v-y-\delta y)+z(w-z-\delta z)\right]=0, \tag{1}$$

$$P_2: \ \delta\left[x(u-x+\delta x)+y(v-y+\delta y)+z(w-z+\delta z)\right]=0. \tag{2}$$

Now we shall take indicatrix $I(P)$ at point $P(x, y, z)$ as two planes $P_i, i=1,2$ having normal PF_i, respectively and pass through F_i, respectively.

Let $(\dot{x}, \dot{y}, \dot{z})$ be the coordinates in the tangent planes at P, induced from (x, y, z), i.e., $\dot{x} = u - x$, $\dot{y} = v - y$, $\dot{z} = w - z$. Then the above equations of planes are rewritten as:

$$x\dot{x}+y\dot{y}+z\dot{z}-\delta\left(x^2+y^2+z^2\right)=0, \tag{3}$$

$$x\dot{x}+y\dot{y}+z\dot{z}+\delta\left(x^2+y^2+z^2\right)=0. \tag{4}$$

It is clear that [2] if the indicatrix is given by an equation $F(x, y, z, \dot{x}, \dot{y}, \dot{z}) = 0$, the fundamental function $L(x, y, z, \dot{x}, \dot{y}, \dot{z})$ is defined by $F\left(x, y, z, \frac{\dot{x}}{L}, \frac{\dot{y}}{L}, \frac{\dot{z}}{L}\right)=0$. Applying this method we get the following fundamental function with respect to which the indicatrix is the above pair of planes:

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{x\dot{x}+y\dot{y}+z\dot{z}}{\delta\left(x^2+y^2+z^2\right)}, \tag{5}$$

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = -\frac{x\dot{x}+y\dot{y}+z\dot{z}}{\delta\left(x^2+y^2+z^2\right)}. \tag{6}$$

Thus we have the following Theorem 2.1.

Theorem 2.1. The metric function of three-dimensional Finsler Space $\mathbb{R}^3$, where indicatrix of any arbitrary point $P(x,y,z)$ is pair of planes is given by

$$L(x,y,z,\dot{x},\dot{y},\dot{z}) = \frac{|x\dot{x}+y\dot{y}+z\dot{z}|}{\delta\left(x^2+y^2+z^2\right)}, \tag{7}$$

where $\delta(x,y,z)$ is positive valued function defined on $\mathbb{R}^3-\{0\}$.

Next we deal with geodesics in $\mathbb{R}^3$ with fundamental function (7). Using coordinate transformation $x=\rho\sin\theta\cos\phi$, $y=\rho\sin\theta\sin\phi$, $z=\rho\cos\phi$ in Eq. (7) where (ρ,θ,ϕ) are spherical polar coordinates in $\mathbb{R}^3$, we have

$$L = \frac{\dot{\rho}}{\delta_\rho}, \tag{8}$$

where $\delta(x,y,z)\cong\delta(\rho,\theta,\phi)$. Thus Eq. (8) gives

$$\left.\begin{aligned}
\frac{\partial L}{\partial \rho} &= \frac{-\rho}{\delta_\rho}\left[\frac{\delta_\rho}{\delta}+\frac{1}{\rho}\right],\\
\frac{\partial L}{\partial \dot{\rho}} &= \frac{1}{\delta_\rho},\\
\frac{\partial L}{\partial \theta} &= -\frac{\dot{\rho}}{\delta_\rho^{\ 2}}\delta\theta,\\
\frac{\partial L}{\partial \dot{\theta}} &= 0,\\
\frac{\partial L}{\partial \phi} &= -\frac{\dot{\rho}}{\delta_\rho^{\ 2}}\delta\phi,\\
\frac{\partial L}{\partial \dot{\phi}} &= 0.
\end{aligned}\right\}, \tag{9}$$

where $\delta_\rho = \frac{\partial\delta}{\partial\rho}$. Hence for any curve $(\rho(t),\theta(t),\phi(t))$ Euler's equation for extremum of curve is given by

$$\frac{\partial L}{\partial \rho}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right)=0,\ \frac{\partial L}{\partial \theta}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)=0,\ \frac{\partial L}{\partial \phi}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)=0. \tag{10}$$

Eq. (10) gives

$$\frac{d\rho}{dt} = \dot{\rho}\delta_\rho, \qquad \delta\theta=0,\ \delta\phi=0, \tag{11}$$

which shows $\delta(\rho,\theta,\phi)$ is only function of ρ. Thus if we take $\delta(\rho,\theta,\phi)=\rho$, then the extremal of $\int_{t=t_1}^{t} L(\rho,\theta,\phi,\dot{\rho},\dot{\theta},\dot{\phi})dt$ gives $-\frac{1}{\rho}$ = constant or ρ = constant, which represents a sphere. Thus we have the following Theorem 2.2.

Theorem 2.2. The geodesics in three-dimensional Finsler Space $\mathbb{R}^3$ is a sphere if indicatrix at any arbitrary point $P(x,y,z)$ is pair of planes provided $\delta(x,y,z)$ is a positive homogenous function.

Euclidean length of OP is $OP=\sqrt{x^2+y^2+z^2}=\rho$. If we take δ as a constant, in particular $\delta=1$, then extremal of length integral $\int_{t=t_1}^{t} L(\rho,\theta,\phi,\dot{\rho},\dot{\theta},\dot{\phi})dt$ is $\log\rho$ = constant, which is a logarithmic spiral. Conversely if geodesics with fundamental function given by Eq. (7) is a logarithmic spiral (not shown here), then $\log\rho$ = constant, which gives ρ as constant and hence Eq. (11) gives $\delta(\rho,\theta,\phi)$ = constant. Thus we have the following Theorem 2.3.

Theorem 2.3. Any geodesic of three-dimensional Finsler Space $\mathbb{R}^3$ with indicatrix of an arbitrary point is a pair of planes is logarithmic spiral if and only if density $\delta(x,y,z)$ is constant.

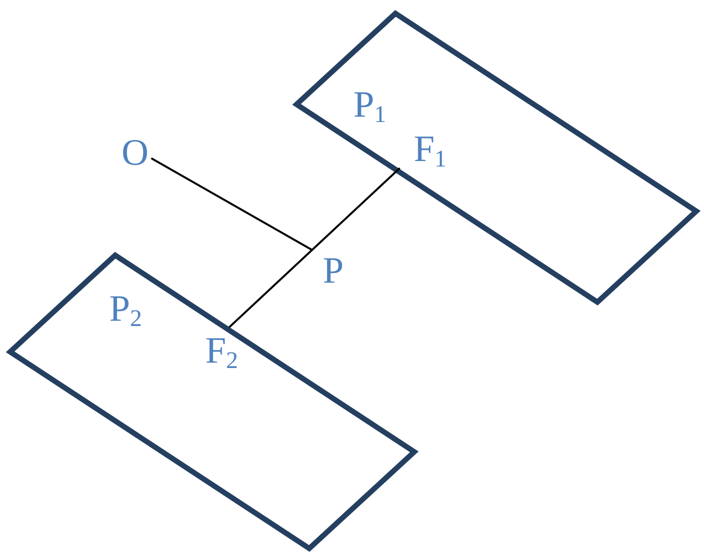

Fig. 1. The indicatrix as a pair of planes in three dimensional Finsler Space.

Further, from Eq. (7) we observe that $l^i=\frac{y^i}{L}$ gives $l^i=\frac{1}{\beta}(\dot{x},\dot{y},\dot{z})$ where $\beta=\frac{x\dot{x}+y\dot{y}+z\dot{z}}{\delta(x^2+y^2+z^2)}$ and $l_i=\frac{\partial L}{\partial y^i}$ gives $l_i=(\lambda x,\lambda y,\lambda z)$ where $\lambda=\frac{1}{\delta(x^2,y^2,z^2)}$.

Now, L is differential one-form metric therefore from $g_{ij}=\frac{1}{2}\frac{\partial^2 L^2}{\partial \dot{x}^i \partial \dot{x}^j}$ gives g_{ij} is independent from $(\dot{x}^i)$ and $C_{ijk}=\frac{1}{2}\frac{\partial g_{ij}}{\partial \dot{x}^k}=0=\frac{1}{2}\frac{\partial g_{ij}}{\partial \dot{x}^k}=0$ using Eq. (7). Thus we have the following Corollary 2.1.

Corollary 2.1. Three-dimensional Finsler Space becomes a three-dimensional Riemannian Space if indicatrix of any arbitrary point is a pair of planes.

3. Indicatrix at any arbitrary point as right circular cylinder

Similar to Section 2, let us consider orthonormal coordinate system (x,y,z) with origin O in three-dimensional Finsler Space $\mathbb{R}^3$ and let $\delta(x,y,z)$ is a positive valued function defined on $\pi=\mathbb{R}^3-\{0\}$. Then we shall define a Finsler metric on π, where indicatrix of arbitrary point $P(x,y,z)$ is right circular cylinder.

For arbitrary point $P(x,y,z)$ of π (refer Fig. 2) we take two points F_1, F_2 and two circles C_1, C_2 such that

(i) PF_1 and PF_2 are orthogonal to OP and their Euclidean lengths are equal to $\delta(x,y,z)$- fold of OP.

(ii) C_1 and C_2 has centers F_1 and F_2, respectively and they have same radius equal to $\delta(x,y,z)$.

(iii) Height of cylinder is equal to 2δ.

We choose F_1 and F_2 as

$F_1\left(x+\delta(x,y,z)x, y+\delta(x,y,z)y, z+\delta(x,y,z)z\right)$ and

$F_2\left(x-\delta(x,y,z)x, y-\delta(x,y,z)y, z-\delta(x,y,z)z\right)$, respectively.

Therefore equation of axis of cylinder is

$$\frac{u-x}{x}=\frac{v-y}{y}=\frac{w-z}{z}. \tag{12}$$

Hence equation of right circular cylinder with above axis and with radius δ becomes

$$\left[z(v-y)-y(w-z)\right]^2+\left[x(w-z)-z(u-x)\right]^2+\left[y(u-x)-x(v-y)\right]^2 \\ =\delta^2\left(x^2+y^2+z^2\right). \tag{13}$$

Let us take indicatrix $I(P)$ at $P(x,y,z)$ as right circular cylinder given by Eq. (13). Let $(\dot{x},\dot{y},\dot{z})$ be the coordinates in the tangent planes at P, induced from (x,y,z), i.e., $\dot{x}=u-x$, $\dot{y}=v-y$, $\dot{z}=w-z$. Then Eq. (13) of right circular cylinder is written as

$$(z\dot{y}-y\dot{z})^2+(x\dot{z}-z\dot{x})^2+(y\dot{x}-x\dot{y})^2=\delta^2(x^2+y^2+z^2),$$

which can be rewritten as

$$(x^2+y^2+z^2)(\dot{x}^2+\dot{y}^2+\dot{z}^2)-(x\dot{x}+y\dot{y}+z\dot{z})^2=\delta^2(x^2+y^2+z^2). \tag{14}$$

Now applying Okubo's method [5], i.e., we replace $(\dot{x},\dot{y},\dot{z})$ by $\left(\frac{\dot{x}}{L},\frac{\dot{y}}{L},\frac{\dot{z}}{L}\right)$, we have

$$L=\frac{1}{\delta}\sqrt{(\dot{x}^2+\dot{y}^2+\dot{z}^2)-\frac{(x\dot{x}+y\dot{y}+z\dot{z})^2}{(x^2+y^2+z^2)}}. \tag{15}$$

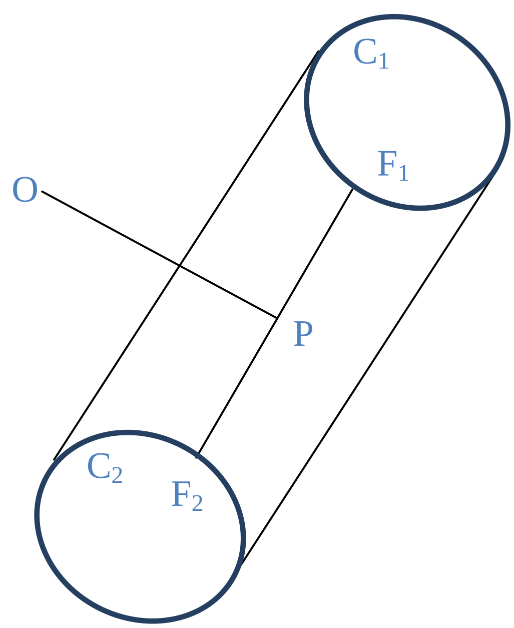

Fig. 2. The indicatrix as a right circular cylinder in three dimensional Finsler Space.

Using coordinate transformation, $x=\rho\sin\theta\cos\phi$, $y=\rho\sin\theta\sin\phi$, $z=\rho\cos\phi$, we have $\dot{x}=\dot{\rho}$, $\dot{y}=\rho\dot{\theta}$, $\dot{z}=\rho\sin\theta\dot{\phi}$. Thus the expression for L given in Eq. (15) becomes

$$L=\frac{\rho}{\delta}\sqrt{\dot{\theta}^2+\sin^2\theta\dot{\phi}^2}, \tag{16}$$

where $\delta(x,y,z)\cong\delta(\rho,\theta,\phi)$. Thus we have the following Proposition 3.1.

Proposition 3.1. In three-dimensional Finsler Space if indicatrix at any arbitrary point $P(\rho,\theta,\phi)$ is right circular cylinder with density $\delta(\rho,\theta,\phi)$ of P, then its fundamental function is given by Eq. (16).

Further, we deal with geodesics in the Space whose fundamental function is given by Eq. (16). Consider Euler's equation for extremal of length integral $\int_{t=t_1}^{t} L(\rho,\theta,\phi,\dot{\rho},\dot{\theta},\dot{\phi})dt$ along curve $x^i = x^i(t), i=1,2,3$, which is given by

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = 0, i=1,2,3. \tag{17}$$

Here we take the coordinates (ρ,θ,ϕ) at the point P. Then for the component ρ, Eq. (17) becomes $\frac{\partial L}{\partial \rho} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) = 0$, which in view of Eq. (16) becomes

$$\sqrt{\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2}\,\frac{\partial}{\partial \rho}\left(\frac{\rho}{\delta}\right) = 0. \tag{18}$$

After integration Eq. (18) gives

$$\delta = c\rho, \tag{19}$$

where c is arbitrary constant. Using Eq. (19) into Eq. (17), we get

$$L = c\sqrt{\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2}\,. \tag{20}$$

Now for the component θ, Eq. (17) becomes $\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$, which in view of Eq. (20) becomes

$$\frac{\sin\theta\cos\theta\dot{\phi}^2}{\sqrt{\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2}} - \frac{d}{dt}\left(\frac{\dot{\theta}}{\sqrt{\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2}}\right) = 0. \tag{21}$$

Similarly, for the component ϕ, Euler's equation (17) gives

$$\frac{d}{dt}\left(\frac{\sin^2\theta\dot{\phi}}{\sqrt{\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2}}\right) = 0, \quad \text{which gives} \quad \frac{\sin^2\theta\dot{\phi}}{\sqrt{\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2}} = \text{constant}.$$

Using above result in Eq. (21) we get $\cot\theta\dot{\phi} - \frac{d}{dt}\left(\frac{\dot{\theta}}{\sin^2\theta\dot{\phi}}\right) = 0$, which can be written as

$$d\left(\cos ec^2\theta\frac{d\theta}{d\phi}\right) = \cot\theta d\phi. \tag{22}$$

Putting $\cot\theta = u$ in Eq. (22), we get $\frac{-du}{d\phi} d\left(\frac{du}{d\phi}\right) = u\,du$. On integration, we have $u^2 + \left(\frac{du}{d\phi}\right)^2 = C_1^{\,2}$, where C_1 is an arbitrary constant. This equation may be written as $\frac{du}{\sqrt{C_1^{\,2} - u^2}} = d\phi$, which after integration gives $u = C_1\left(\sin\phi + C_2\right)$, where C_2 is another arbitrary constant.

Taking $C_2 = 0$ and using $u = \sin\theta$, the above equation may be written as $\cot\theta = C_1 \sin\phi$, i.e.,

$$z = cy, \tag{23}$$

which is the plane on which geodesic lies. Since L is invariant under orthogonal change of coordinates (x, y, z), therefore proposition 1 of [10] shows that any geodesic is logarithmic spiral as plane curve and lies on a plane which passes through from origin O. Thus we have the following Theorem 3.1.

Theorem 3.1. Any geodesic of three-dimensional Finsler Space where indicatrix at any arbitrary point is right circular cylinder is logarithmic spiral as a plane curve, where plane possess through origin O.

4. Discussion and conclusions

Hojo [10] considered fundamental metric of three-dimensional Finsler Space given by

$$L = k\sqrt{\left[\frac{\left(\dot{x}^1\right)^2 + \left(\dot{x}^2\right)^2 + \left(\dot{x}^3\right)^2}{\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2}\right]} - h\left[\frac{x^1\dot{x}^1 + x^2\dot{x}^2 + x^3\dot{x}^3}{\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2}\right],$$

$$L = k\left[\frac{\left(\dot{x}^1\right)^2 + \left(\dot{x}^2\right)^2 + \left(\dot{x}^3\right)^2}{\left| x^1\dot{x}^1 + x^2\dot{x}^2 + x^3\dot{x}^3 \right|}\right] - h\left[\frac{x^1\dot{x}^1 + x^2\dot{x}^2 + x^3\dot{x}^3}{\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2}\right],$$

and proved that its geodesics are logarithmic spiral whereas we started with indicatrix at any point $P(x, y, z)$ and use the notion of density $\delta(x, y, z)$ [1], but it does not change the geodesic of the Space. Thus, any geodesic of three-dimensional Finsler Space where indicatrix at any arbitrary point is right circular cylinder, and is a logarithmic spiral as a plane curve.

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